Linear Algebra (18.06) ①

Lecture 1

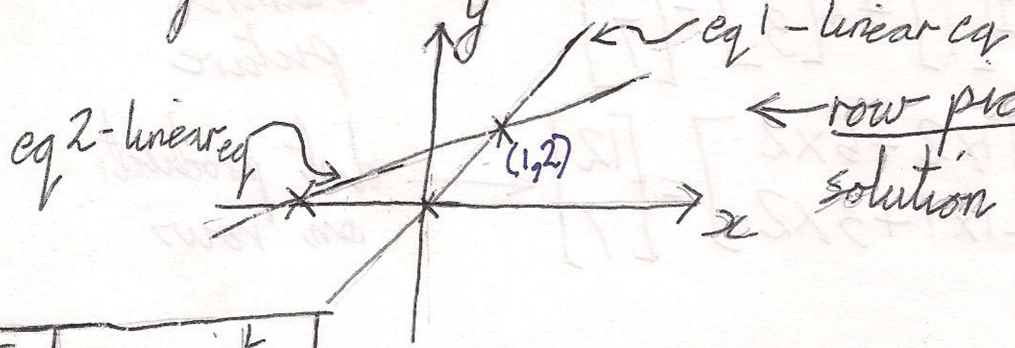
n-linear eq^s, n unknowns → the basic usage (nice case)

Row picture → $2x - y = 0$, $-x + 2y = 3$ → Coefficient matrix

$$\underbrace{\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix}0\\ 3\end{bmatrix}}_{b}$$

vectors

* single column matrices are vectors



← row picture

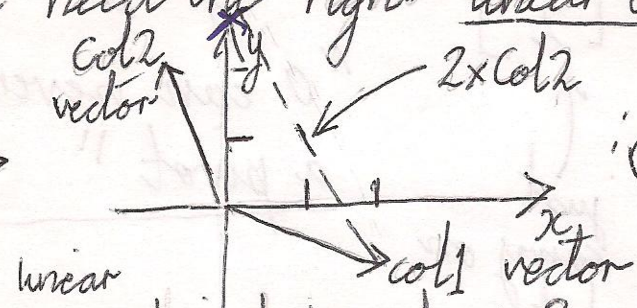
solution $x = 1, y = 2$

Column picture

$$x\begin{bmatrix}2\\ -1\end{bmatrix} + y\begin{bmatrix}-1\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 3\end{bmatrix}$$

we need the right linear combinations of the columns

column picture →



; $x = 1, y = 2$ → $b = \begin{bmatrix}0\\ 3\end{bmatrix}$

All the linear combinations of X & y will fill the whole plane ~~for~~

~~some column combinations~~

→ In three dimensions, using the row picture, ~~2 will mee~~ 3 planes would meet a single point

→ Can I solve $AX = b$ for every b?

* if there is, elimination will find the answer systematically. In other words, do linear combination of the columns fill the 3-D space? (for a 3 dimensional vector)

** Yes you can for a non-singular/invertible matrices

If the 3 vectors happen to lie in the same plane, we cannot get to all b's, just those in that plane. (in column picture)

Multiplication

Matrix X vector

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}\longleftarrow \text{column picture}$$

or

$$=\begin{bmatrix}1\times2+5\times2\\1\times1+3\times2\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}\longleftarrow \text{dot product on rows}$$

Lect 2

elimination

$$\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\12\\2\end{bmatrix}$$

$x+2y+z=2$
$3x+8y+z=12$
$4y+z=2$

"0 can never be a pivot"

minus row 1 from row 2 to knock out x in (2,1)

1st pivot

$$\begin{bmatrix}\boxed{1}&2&1\\0&2&-2\\0&4&1\end{bmatrix}$$

just carry on later

minus row 2 from row 3 to knock out y in (3,2)

$$\begin{bmatrix}\boxed{1}&2&1\\0&\boxed{2}&-2\\0&0&\boxed{5}\end{bmatrix}\begin{matrix}\longleftarrow 2^{nd}\text{ pivot}\\ \longleftarrow 3^{rd}\text{ pivot}\end{matrix}$$

hey! the determinant is multiplications of pivots (10)

Q. How can this fail?

if (1,1) was a 0. But you can always get out of trouble as long as you have a non zero value below in some row (then just exchange the rows)

Failure case

$$\begin{bmatrix}\boxed{1}&2&1\\3&8&1\\0&4&-4\end{bmatrix} \qquad \begin{bmatrix}2\\12\\2\end{bmatrix} \leftarrow$$

lets ignore this, and carry along in the end

$$\begin{bmatrix}\boxed{1}&2&1\\0&\boxed{2}&-2\\0&4&-4\end{bmatrix}$$

$$\begin{bmatrix}\boxed{1}&2&1\\0&\boxed{2}&-2\\0&0&⓪\end{bmatrix}$$

← woops its a zero! a pivot cant be a zero! Not invertible matrix / no sol.

back Substitution

$$\begin{bmatrix}\boxed{1}&2&1\\3&8&1\\0&4&1\end{bmatrix} \qquad \begin{bmatrix}2\\12\\2\end{bmatrix} \leftarrow$$

augmented matrix (same thing to both A & b)

$$\begin{bmatrix}\boxed{1}&2&1\\0&\boxed{2}&-2\\0&4&1\end{bmatrix} \qquad \begin{bmatrix}2\\6\\2\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}\boxed{1}&2&1\\0&\boxed{2}&-2\\0&0&\boxed{5}\end{bmatrix}}_{\underline{U}} \qquad \underset{\underline{c}}{\begin{bmatrix}2\\6\\-10\end{bmatrix}} \longrightarrow \begin{aligned}x+2y+z&=2\\2y-2z&=6\\5z&=-10\end{aligned}$$

system is upper triangular

so $z=-2$

go to row 2 → $y=1$

go to row 1 → $x=1$

Matrices

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

the big picture

matrix x column = column

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} - \\ - \\ - \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \end{bmatrix}$$

row x matrix = row

$$\begin{bmatrix} - & - & - \end{bmatrix}\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{bmatrix} - & - & - \end{bmatrix}$$

what matrix will subtract 3 x row 1 from row 2

E_{21} elementary or elimination matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \boxed{-3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

how to check a particular entry
lets say (2, 3) → multiply and sum up row 2 of matrix 1 & column 3 of matrix 2

→ Step 2 of elimination (Subtract 2 x row 2 from row 3

$$E_{32} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

How to put these E (elementary or elimination) matrices together

$$E_{32}(E_{21}A) = U$$

↖ upper triangular matrix

$$E_{32}(E_{21}A) = U$$

$$(E_{32}E_{21})A = U$$ — (associative law for matrix multiplicat)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A = U$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}$$

but there is a better way to do all of this by:

Inverses

$$\begin{bmatrix} & ? & \\ & & \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

what matrix will give you the original matrix? In other words what matrix when multiplied would give you an identity - (not I get A to U but how to get U to A)

$$\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{=}{=} \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Permutation

Exchange rows 1 & 2

$$\underset{P \leftarrow \text{exchange rows of } I}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Exchange col 1 & 2

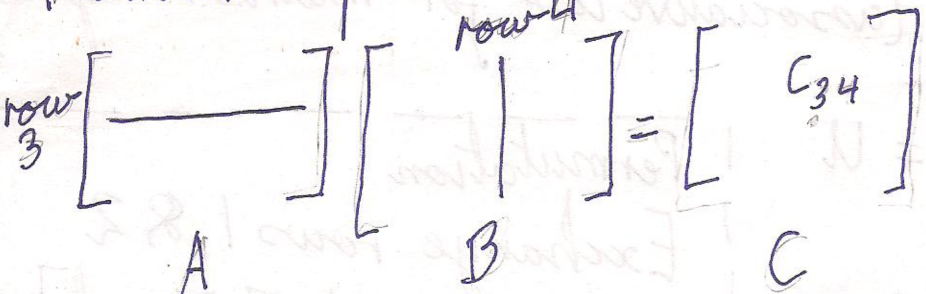
$$\underset{P}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

<u>multiply on left to do row operations // multiply on right to do column operations</u>

Lect 3

Matrix multiplication

①

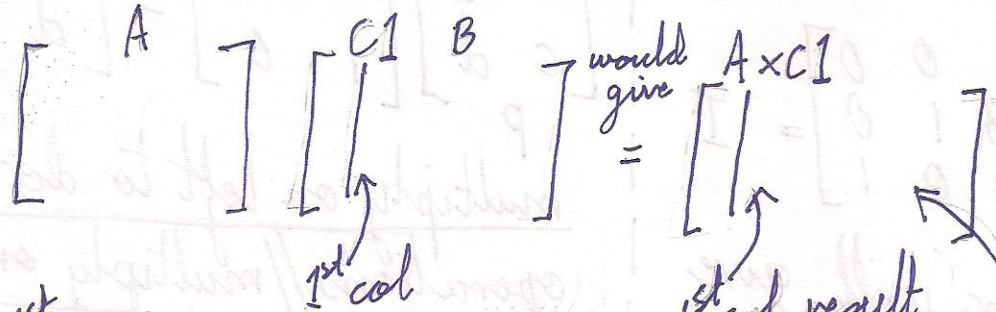


$$c_{34} = \sum_{i=1}^{n} a_{3i} b_{i4} = (\text{row } 3 \text{ of } A).(\text{col } 4 \text{ of } B)$$

When are we allowed to multiply two matrices?

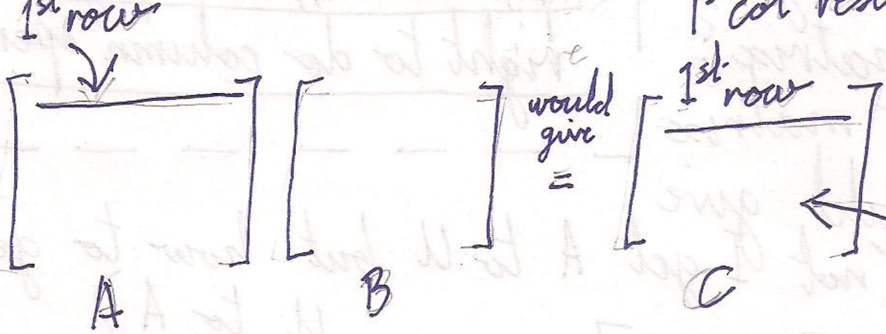
$m \times n$ — $n \times p \longrightarrow m \times p$

M1 M2 Matrix result dimension



②

columns of C are combinations of columns of A



③

rows of C are combinations of rows of B

column of A × row of B ④

$m \times 1$ $\quad$ $1 \times p$

should give a whole $m \times p$ matrix

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

both columns of C are multiples of A

all rows of C are multiples of B

④

AB = sum of (cols of A) x (rows of B)

$$\therefore \begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

Block multiplication ⑤

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ \hline A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{array}\right]$$

Inverses

Not all matrices have inverses

$$\underset{\text{if it exists}}{A^{-1}}A = I = \underset{\text{for square matrices}}{AA^{-1}}$$

(in invertible

, non-singular matrices)

singular matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \leftarrow$$ no determinant

no inverse

2-second reason: both columns are combinations of each other, no multiples of each other ~~and~~ would give I

third reason: you can find a non-zero vector x resulting in $A\underline{x} = 0$ (because if some linear combination of columns results in annihilation, then at least one of the columns doesn't contribute to the matrix)

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\overline{A^{-1}}\,\overline{A}\,\overline{x} = \overline{A^{-1}}\,\overline{0} \; !$$

an invertible matrise

$$\begin{bmatrix}1&2\\3&\boxed{7}\end{bmatrix}\underset{A^{-1}}{\begin{bmatrix}a&c\\b&d\end{bmatrix}}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

any thing other than 6

$A \times$ col j of A^{-1} = column j of I

Gauss-Jordan (solve 2 equations at once)

$$\underset{A}{\begin{bmatrix}1&2\\3&7\end{bmatrix}}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}1&2\\3&7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cc|cc}1&2&1&0\\3&7&0&1\end{array}\right]$$

augmented matrix

turn left hand side into identity, and right hand side will give the inverse

by elimination

$$\overset{A}{\left[\begin{array}{cc|cc}1&2&1&0\\3&7&0&1\end{array}\right]}$$

$$\rightarrow \left[\begin{array}{cc|cc}1&2&1&0\\0&1&-3&1\end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc}1&0&7&-2\\0&1&-3&1\end{array}\right]$$

check

$$\begin{bmatrix}1&2\\3&7\end{bmatrix}\begin{bmatrix}7&-2\\-3&1\end{bmatrix}=\begin{bmatrix}7+(-6)&-2+2\\21-21&-6+7\end{bmatrix}=I$$

so $EA = I$ tells us that $E = A^{-1}$

E → comb. of elimination

Lect 4

Inverse of a product

Singular Matrices have a zero determinant

$$\underline{(AB)}\underline{(B^{-1}A^{-1})} = \underline{I}$$

$$AA^{-1} = I$$

$$(A^{-1})^{T}A^{T} = I$$

↑ inverse of (A^{T}) which is $(A^{T})^{-1}$

→ Transposing and inversing can be done in any order/Inverse of A^{T} is the transpose of (A^{-1})

→ $\underline{A = LU}$ is the most basic <u>matrix factorization</u>

$$\begin{array}{cccc} E_{21} & A & & U \\ \begin{bmatrix}1 & 0\\ -4 & 1\end{bmatrix} & \begin{bmatrix}2 & 1\\ 8 & 7\end{bmatrix} & = & \begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}\end{array}$$

$$\begin{array}{cccc} A & & L & U \\ \begin{bmatrix}2 & 1\\ 8 & 7\end{bmatrix} & = & \begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix} & \begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}\end{array}$$

L ⇒ lower triangular
U ⇒ upper triangular

$$L = E_{21}^{-1}$$

U is the upper triangular, L is the lower triangular

U has pivots on the diagonal, L has 1's on the diagonal

$$\begin{array}{ccccc} A & & L & Z & U \\ \begin{bmatrix}2 & 1\\ 8 & 7\end{bmatrix} & = & \begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix} & \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix} & \begin{bmatrix}1 & \frac{1}{2}\\ 0 & 1\end{bmatrix}\end{array}$$

→ for a 3 x 3 matrix

$$E_{32}E_{31}E_{21}A = U$$ (no row exchanges)

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU$$

E_{xy} is read as elimination matrix to get 0 in row x, col y suppose E_{32} $\quad E_{31} = I \quad$ E_{21} $\quad E$ (left of A) $\rightarrow EA = U$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

(inverses in opp. order) $\overset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} \rightarrow A = LU$

$\boxed{A = LU}$

if no row exchanges, the elimination matrix multiples go directly into L

→ How many operations on $n \times n$ matrix A would take to do elimination?

change n-1 rows
each row has n multiplications
+ subtractions

first step $\begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \begin{bmatrix} - & - & - & - \\ 0 & - & - & - \\ \vdots & - & - & - \\ \downarrow & - & - & - \end{bmatrix} \rightarrow (n-1)2n$ steps

Hence total steps would be $\sum_{n=2}^{100} (n-1)2n \approx \frac{2}{3} n^3 < n^3$
for a 100×100 matrix

→ **What in case of row exchanges?**

Permutation row matrices (using the I)

$P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{23} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad P_{12} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

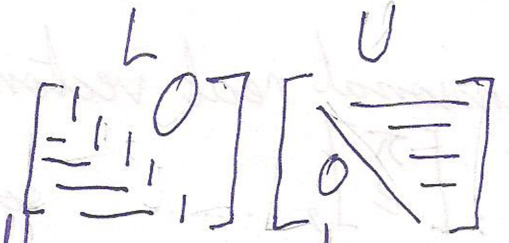
where $\underline{P^T = P^{-1}}$

$P_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{\substack{13 \\ 12}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{\substack{13 \\ 23}} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Lect 5

Permutations P: execute row exchanges

$A =$



← without row exchanges – $P = I \Rightarrow PA = LU$

but with row exchanges $PA = LU$ // any invertible A

↗ permutation of I's row matrix

$P =$ identity matrix with re-ordered rows

for an $n \times n$ identity matrix

$\boxed{P^{-1} = P^T}$ (n! possible matrices)

Matrix transposes $\longrightarrow \begin{bmatrix}1 & 2\\ 2 & 3\\ 4 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\ 2 & 3 & 1\end{bmatrix}$

a matrix $a \times b$ will have a $b \times a$ size matrix transpose

$(A^T)_{ij} = A_{ji}$

Symmetric matrices $A^T = A \longrightarrow \begin{bmatrix}3 & 1 & 8\\ 1 & 7 & 6\\ 8 & 6 & 9\end{bmatrix}$

if $R = \begin{bmatrix}1 & 3\\ 2 & 3\\ 4 & 1\end{bmatrix}$ $R^T = \begin{bmatrix}1 & 2 & 4\\ 3 & 3 & 1\end{bmatrix}$

RR^T is always symmetric

$$RR^T = \begin{bmatrix}1 & 3\\ 2 & 3\\ 4 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 4\\ 3 & 3 & 1\end{bmatrix} = \begin{bmatrix}10 & 11 & 7\\ 11 & 13 & 11\\ 7 & 11 & 17\end{bmatrix}$$

$(R^TR)^T = R^TR$ ← order gets reversed $(R^TR)^T = R^TR^{TT}$

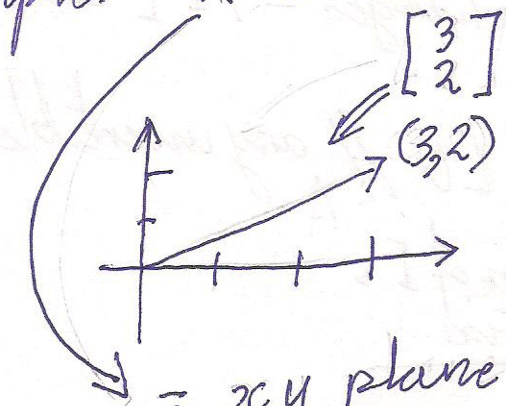
this shows why R^TR is symmetric

Vector Spaces

spaces means a bunch of vectors which allow linear combinations

Examples R^2 = all 2-dimensional real vectors like

$$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}, \ldots\ldots \text{ so on}$$



The whole plane is R^2

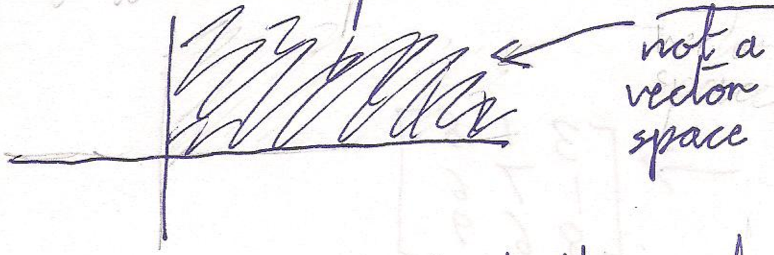
= xy plane (it has all real 2D vectors)

Similarly R^3 = all 3-dimensional real vectors

$$\begin{bmatrix}3\\2\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \ldots \text{ so on}$$

R^n = all column vectors with n components

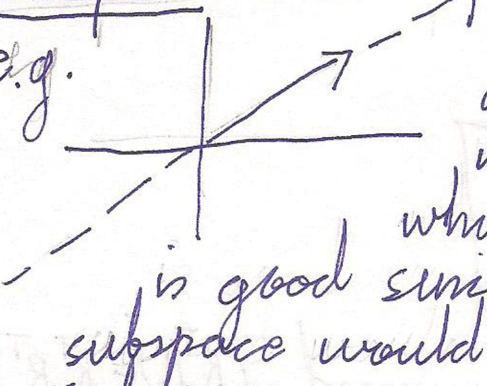
not a vector space since its not closed in multiplication of vectors with scalars



Subspace

subset of the vector space inside R^n

e.g.



you can't have a vector space without a whole line in R^2 since its multiplicat with a scalar should produce a vector which is included in the subspaces. A line is good since, an addition of vectors within a subspace would result in a vector already in that subspace.

A subspace in R^2 will only be possible if the line passes through zero vector

Subspaces of R^2

① all of R^2

② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

③ the zero vector alone (Z)

Subspaces R^3

① all R^3

② plane through the origin

③ line through the origin

④ the zero vector

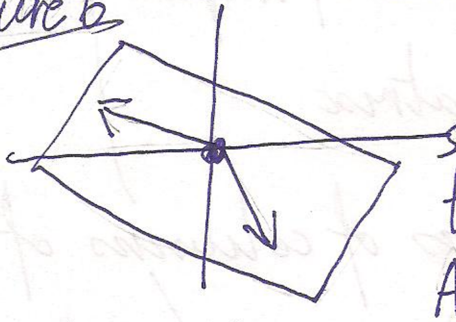
How do subspaces come out of matrices?

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ columns in R^3 will have to take all their linear combinations to form a subspace

.... this is called a column space $C(A)$

key idea, which is a plane in 3D
through the origin

Lecture 6



A plane through $(0,0,0)$ is a subspace of R^3 which is made by two vectors

A line is also a subspace through $(0,0,0)$ which contains a vector and its multiples. All subspaces should contain the zero vector

Q- If we have 2 subspaces. P (a plane) and L (a line), is $\underset{\text{union}}{P \cup L}$ = all vectors in P or L is a subspace or not?

No because if I add a vector from L and a vector from P, we will go outside $P \cup L$

Q- Is $\underset{\text{intersection}}{P \cap L}$ = all vectors in both P and L is a subspace

Yes because it contains only the zero vector, the intersection point of the plane and line

→ For any subspace S and T, their intersection $S \cap T$ is also always a subspace

If we take v && w vectors, it would imply that both of them are in S and both are in T, hence v+w is contained $S \cap T$. Req 2 of any scalar multiple is also in $S \cap T$

→ Vector space requirements: v+w and cv are in the space if all combinations cv+dw are in space

Column space of A → C(A)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$ is a subspace of R^4 because A is 4x3 matrix

i.e do linear combinations of column of A span ~~R~~ all of R^4

C(A) = all linear combinations of columns of A which makes a subspace

Q - does Ax = b has a solution for every b? No

because Ax is 4 equations and 3 unknowns. i.e we can't have solutions for the whole subspace of R^4 with a linear comb. of 3 vectors

→ but which vectors b allow this system to be solved??

for one if b is (0, 0, 0, 0)

+ all of the individual columns like (1, 2, 3, 4)

i.e I can solve Ax = b exactly when b is in the column space of A - C(A)

Q-are the columns of A independent i.e. does every new column contributing new?

No because column 3 is just a column that is a combination of column 1 & 2. I could even throw-out column 1 or 2. But for our convention 1 & 2 become the pivot columns

Hence columns (column space) of A form a two dimensional subspace of R^4

Null space $N(A)$ of A contains all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$ i.e.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Null space is in R^3

→ In $Ax = B$, where A is an $m \times n$ matrix, column spaces are in R^m and null spaces are in R^n

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

→ one solution is $(0,0,0)$

→ $(1,1,-1)$

in other words $(-c,-c,c)$, i.e. and $c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ for any c hence $N(A)$ is a line through $(-1,-1,1)$ inside R^3. Hence we have a subspace

→ check that $N(A)$ i.e $Ax=0$ always gives a subspace

If $Av = 0$ & $Aw = 0$ then $A(v+w) = 0$ since $Av + Aw = 0$

Why $A(v+w) = Av + Aw$?

$$A\begin{bmatrix}v_1+w_1\\ v_2+w_2\\ v_3+w_3\end{bmatrix} = \begin{bmatrix}a_{11}(v_1+w_1)+a_{12}(v_2+w_2)+a_{13}(v_3+w_3)\\ \vdots \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix}(a_{11}v_1+a_{12}v_2+a_{13}v_3)+(a_{11}w_1+a_{12}w_2+a_{13}w_3)\\ \vdots \\ \\ \end{bmatrix}$$

$$= A\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} + A\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}$$

$$Ax = \begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}$$

the solutions of x do ~~they~~ form a subspace?
i.e. not $N(A)$?

No because x cannot be (0, 0, 0). Subspaces better solve $Ax = 0$. (the solution is a line (plane?) that doesn't go through the origin)

Lecture 7

How to compute (what is the algo) $Ax = 0$

$$A = \begin{bmatrix}\boxed{1} & 2 & 2 & 2\\ 2 & 4 & 6 & 8\\ 3 & 6 & 8 & 10\end{bmatrix} \leftarrow$$ 2nd col and 3rd row is not independent

elimination does NOT change the null space but changes the column space

doing elimination on it (even if 0s in pivots)

$$\begin{bmatrix}\boxed{1} & 2 & 2 & 2\\ 0 & 0 & 2 & 4\\ 0 & \underline{0} & 2 & 4\end{bmatrix}$$ this shows the 2nd column is ~~in~~dependent on the 1st

We continue elimination, and select a pivot in the third column

$$\begin{bmatrix}\boxed{1} & 2 & 2 & 2\\ 0 & 0 & \boxed{2} & 4\\ 0 & 0 & 0 & 0\end{bmatrix} \leftarrow$$ which shows that the 3rd row is dependent on 2nd & 1st

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

echelon form (staircase form)

The <u>rank</u> if a matrix is the number of <u>pivots</u>

→ in the above case rank is 2

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free cols are free because we can choose any no. arbitrarily x_i for those cols and then choose an x for pivot cols

pivot cols   free cols

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \quad \text{——— (1)}$$

$$2x_3 + 4x_4 = 0 \quad \text{——— (2)}$$

so choosing arbitrarily $x = \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ looking at eq (2)

to get a sol. in N(A)

hence $x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ which is a line in 4D

again choosing arbitrarily $x = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

<u>special sols.</u>

<u>for computing null space</u>

hence x is also $= d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$$N(A) = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The null space exactly contains all comb. of free variable solutions

∴ → if r is rank, n is the no. of cols, there are $n-r$ free solutions i.e. nullspace would be all combinations of $n-r$ solutions

Reduced row echelon form - R (more cleanup on echelon form)

$$U = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row echelon form has 0's above & below the pivot

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \overset{\text{MATLAB}}{\text{rref}(A)}$$

notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity matrix I in pivot rows/cols

$\therefore Ax = 0 \rightarrow Ux = 0$ & $Rx = 0$

pivot cols $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and free cols $\begin{bmatrix} 2 & -2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$

I F

the clean-up is so good, you can read off the special solution directly for the N(A) $\rightarrow c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

the $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ appears in the x_2 and x_4 rows (free rows)

and $-\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$ appears in the x_1 and x_3 (pivot rows)

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r \text{ pivot rows}$$

r pivot cols

n−r free cols

$$\underset{\text{null space matrix}}{RN} = 0 \text{ hence } N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

for our example

$$R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to adjust col. exchg

$$N = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$$

Another e.g.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \quad \text{its rref} \quad \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

rank is 2 hence 1 special sol

null space

$$x = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$ its a line.

basis of null space

$$\text{rref} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I

F

$$x = c \begin{bmatrix} -F \\ I \end{bmatrix}$$

N

$$\boxed{x_{pivot} = -F x_{free}}$$

Lecture 8

Solve linear equation if there is a solution $\Rightarrow$ $\underline{Ax=b}$

$$\underset{A}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}} x = b$$

$r_1 + r_2$

How to deal with b? Form augmented matrix $[A\ b]$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

pivot cols

Note that this is just an example $b = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ $\Longleftarrow$ $\therefore\ 0 = b_3 - b_2 - b_1$ hence $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is OK!

Solvability Condition on b

$Ax=b$ solvable if when b is in column space $C(A)$

If a combination of the rows of A gives a zero row (like in the above example) then the same combination of entries b must give 0 i.e. why $b_3 - b_2 - b_1 = 0$ in the above e.g.

To find the complete sol. to $Ax=b$

① ~~a~~ find a particular x_p $\longrightarrow$ Set all free variables to zero then solve $Ax=b$ for pivot variables

ⓐ hence set $x_2 = 0$ & $x_4 = 0$

ⓑ ∴ $x_1 + 2x_3 = 1$ $\quad x_1 = -2$

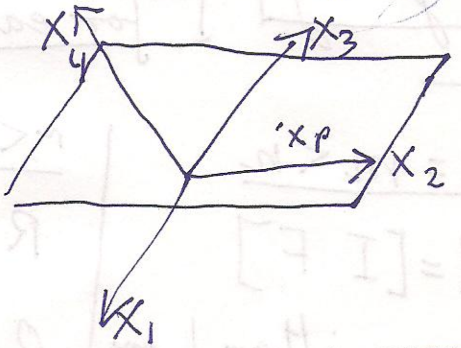
$2x_3 = 3$ $\quad x_3 = \frac{3}{2} \Rightarrow x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

② add any solution from the null-space to x found above:

$$\begin{matrix} Ax_p = b \\ Ax_n = 0 \end{matrix} \Longrightarrow A(x_p + x_n) = b$$

$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Plot all solutions x in R^4



its a plane shifted from the origin to x_p

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m by n matrix A of rank r (we know $r \leq m$, $r \leq n$)

Full column rank $r = n$ means there is a pivot in every column & no free variables

∴ N(A) will only have the zero vector (since no free vars)

& x is $x = x_p$ if there is any at all

e.g. $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \longrightarrow \text{rank } 2 \Longrightarrow R_{rref} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

in this case there will either be 0 solutions or only 1 solution for each b

Full row rank $r = m$ means that every row has a pivot

we can solve $Ax = b$ for **every b**

free variables $n - r$

e.g. $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \longrightarrow$ rank 2 $\quad R = \begin{bmatrix} 1 & 0 & \begin{pmatrix} w & s \\ t & r \end{pmatrix} \\ 0 & 1 & \end{bmatrix} = \begin{bmatrix} 1 & 0 & F \\ 0 & 1 & \end{bmatrix}$

any no.s

$r = m = n$ square matrix

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ its an invertible matrix

$\boxed{R = I}$ $\boxed{N(A) = 0}$

$\boxed{b \text{ can be anything}}$ $\boxed{\underline{1} \text{ solution for each } b}$

Summary

$r = m = n$	$r = \overset{n}{\cancel{m}} < m$	$r = m < n$	$r < m \quad r < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = [I \ F]$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 solution to $Ax = b$	0 or 1 solution	always ~~either 1 or~~ inf. solutions	0 or inf. solutions
	full col. rank	full row rank	

Lecture 9

Suppose A is m by n with $m < n$ (more unknowns than eq.s)

Then there are non-zero solutions to $Ax = 0$

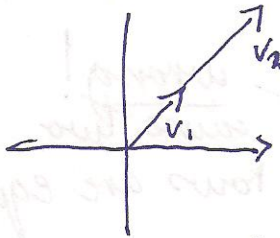
Reason: there will be atleast 1 free variables !!

crucial point in independence

Independence

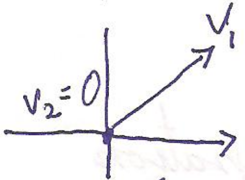
Vectors $x_1, x_2 \ldots\ldots x_n$ are linearly independent if no combination gives a zero vector (except the zero vector)

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n \neq 0 \quad \text{except all } c_i = 0$$



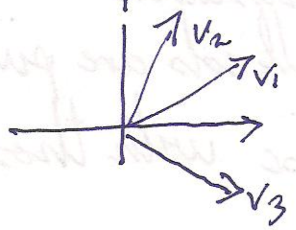
v_1 & v_2 are dependent

$$2v_1 - v_2 = 0$$



v_1 & v_2 are dependent

$$0v_1 + cv_2 = 0$$



if v_1 & v_2 are independent, v_3 will be dependent

$$A = \begin{array}{ccc} v_1 & v_2 & v_3 \\ \left[\begin{matrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{matrix}\right. & & \left.\right] \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since $m < n$ then there will be non-zero solutions

Repeat when $v_1 \ldots, v_n$ are columns of A

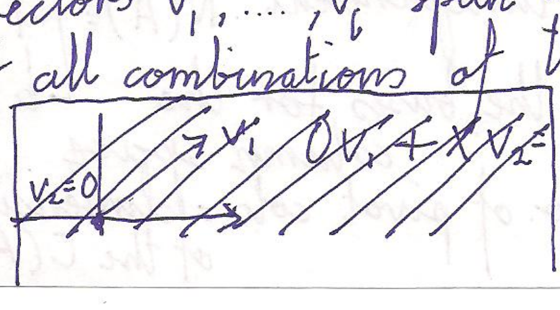
- They are independent if the nullspace of A is only the zero-vector
- They are dependent if $Ac = 0$ for some non-zero c

→ rank $= n$, no free variables, $N(A) = \{0\}$

→ rank $< n$,, ye free variables, $N(A) = \{0) \cup$ some other vectors

What does it mean for some vectors to span a space?

vectors $v_1, \ldots, v_l$ span a space means the space consists of all combinations of those vectors.



Basis for a space is a sequence of vectors $v_1, v_2, \dots v_d$ with 2 properties:

(1) they are independent

(2) they span the space

Example:

Space is $\mathbb{R}^3$

One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ | Another basis $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}1+2\\1+2\\8\end{bmatrix}$ ← wrong! cause two rows are equal

How to check if the ~~n~~ vectors form a basis:

(1) take vector columns, put in matrix

(2) bring it down to reduced form with elimination

(3) check if there are no free variables i.e. all cols are pivot cols.

n vectors give basis if The $n \times n$ matrix with those columns is invertible

So $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ will span a 2D plane themselves

Basis is not unique. Any 3×3 invertible matrix will give the Basis for $\mathbb{R}^3$

Space $\mathbb{R}^n$ will have n basis vectors: ~~every~~ given a ~~ba~~ space every basis has the same number of vectors. That number is the dimension of the space

Example: space is $C(A)$

$$\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}$$

$v_1 \; v_2 \; v_3 \; v_4$

→ they span the column space by def.

→ they are not independent. $N(A) = \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$

→ v_1 & v_2 give the bases for the column space

→ rank$\overset{(A)}{=}$ 2 ⇐ no. of pivot cols = dimension of the $C(A)$

→ another basis v_1, v_3 // v_2, v_4 // v_3, v_4 // ......

v_1+v_4 , $v_1+v_2+v_3+v_4$ also form a basis

i.e. if we take any two combinations, they are independent they will always span the space C(A).

→ dimension of N(A) → $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ by taking combinations of the free variables

dim N(A) = # free variables $= n-r$

Lecture 10

4 subspaces → column space C(A) in $\mathbb{R}^m$

null space N(A) in $\mathbb{R}^n$

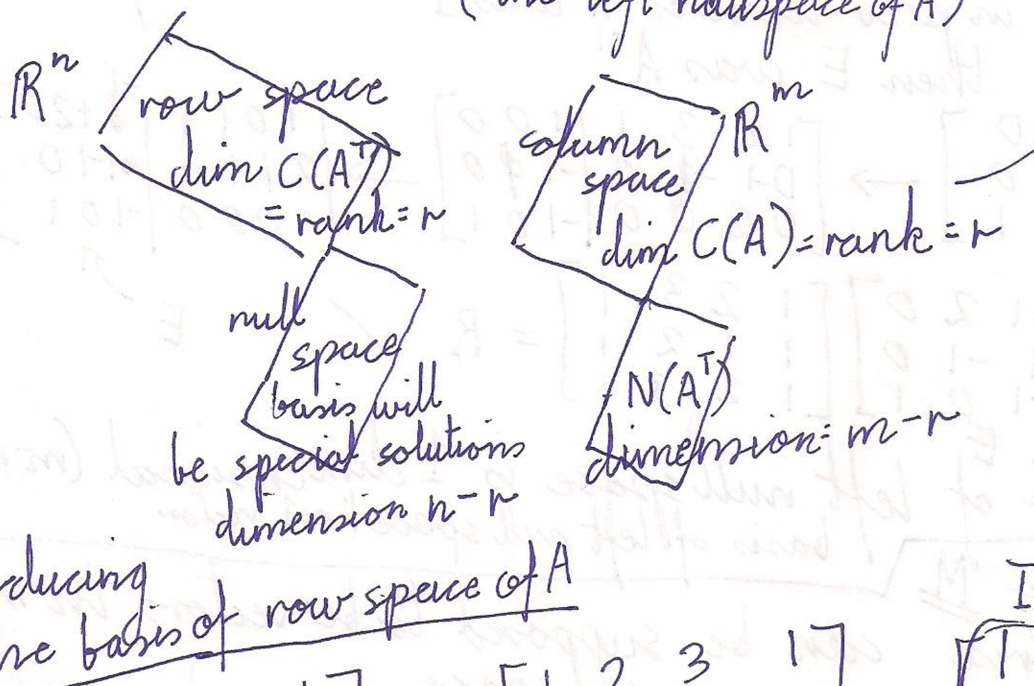
row space = all comb. of rows in $\mathbb{R}^n$

= all comb. of cols of $A^T = C(A^T)$

nullspace of $A^T = N(A^T)$ in $\mathbb{R}^m$

(the left nullspace of A)

when A is m×n



→ pivot col. C(A) dimension r

Producing the basis of row space of A

$$A=\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}\rightarrow\begin{bmatrix}1&2&3&1\\0&-1&-1&0\\0&0&0&0\end{bmatrix}\rightarrow\underset{R}{\begin{bmatrix}\overbrace{\begin{matrix}1&0\\0&1\end{matrix}}^{I}&\overbrace{\begin{matrix}1&1\\1&0\end{matrix}}^{F}\\\begin{matrix}0&0\end{matrix}&\begin{matrix}0&0\end{matrix}\end{bmatrix}}$$

C(R) ≠ C(A) different column space but same row space

basis of row space of A is the first ~~two~~ r rows of R → $\begin{bmatrix}1&0&1&1\\0&1&1&0\end{bmatrix}$

Row space doesn't change by elimination

The 4th space : $N(A^T)$

$A^T y = 0$ then y is in the null space of A^T

$\Rightarrow y^T A = 0^T$

$\begin{bmatrix} & y^T & \end{bmatrix} \begin{bmatrix} \\ A \\ \\ \end{bmatrix} = 0^T$ since its multiplying from the left side that is why it is called left nullspace

doing Gauss jordan $\xrightarrow[rref]{} [A_{m\times n} \; I_{m\times m}] \rightarrow [R_{m\times n} \; E_{m\times m}]$

this will contain a record of what we did

$\underline{\underline{EA = R}}$

in prev lectures R was I then E was A^{-1}

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & -1 & +2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

E

to check $EA = \underbrace{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}}_{A} = R$ ✓

last row is the basis of $N(A^T)$

$\therefore$ dimension of left null space is 1 dimensional $(m-r = 3-2)$

basis of left null space has 1 vector

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new vector space $\underline{\underline{M}}$

All 3x3 matrix can be supposed to be vectors in my space

since matrices also follow the rules of the vector space

$A+B$, cA (not AB for now)

So what are the subspaces of M || upper triangular || symmetric matrices || dim is 3

intersection

|| diagonal matrices ||

So for a symmetric matrix subspace

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&7\end{bmatrix}$$ form the basis for the symmetric matrix subspace

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Lecture 11

We looked at matrices 3x3 as part of a vector space to prove the point that anything that follows the rules (addition & scalar matriles) can be called a vector space

Basis for M = all 3X3 matrices → 9 dimension

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{}$$

same as a 9 dimension vector except they are written in a square form

3 of these are the basis of symmetric matrices

dimension of 3x3 symmetric matrices = 6

$$\begin{bmatrix}a&d&e\\d&b&f\\e&f&c\end{bmatrix}$$

What about Symmetric ∩ Upper triangular (diagonal) ?

$\dim(S \cap U) = 3$

Why is it of no use to talk about $S \cup U$?

because its not a subspace

Then what about their sum $S+U$ = any element of S + any element of U

= all 3x3 matrices !!

$\dim(S+U) = 9$

$\dim S = 6 \quad \dim U = 6 \implies \| \underbrace{\dim S}_{6} + \underbrace{\dim U}_{6} = \underbrace{\dim(S \cap U)}_{3} + \underbrace{\dim(S+U)}_{9}$

Another example of vector space

differential eq.

$$\frac{d^2y}{dx^2} + y = 0 \rightarrow \text{solutions} \rightarrow y = \cos x,\ \sin x,\ \cancel{e^{ix}}$$

dont need for complete sol.

i.e. the null space that is the sol. space

complete solution

$$y = c_1 \cos x + c_2 \sin x$$

the basis of this subspace → $\underline{\cos x}$ & $\underline{\sin x}$

dimension of this solution space $= 2 \leftarrow$ (because of $\frac{d^2y}{dx^2}$)

another basis would be $\rightarrow e^{ix}, e^{-ix}$

Rank one matrices

$$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

the first row would be the basis of row space

the first col would be the basis of col. space

$$= \overset{\text{pivot col.}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \overset{\text{pivot row}}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}$$

All rank 1 matrices

$$A = \underset{\text{1 col}}{u}\ \underset{\text{1 row}}{v^T}$$

<u>They are the building blocks of all matrices</u>

<u>All matrices can be broken down to r rank 1 matrices</u>

→ ~~all~~ A 5x17 rank 4 matrix can be broken down to 4 rank matrix

M = all 5x17 matrices of rank 4 matrices ... do they form a ~~sups~~ subspace? <u>No</u>

→ or even rank 1 matrices? <u>No</u>

In $\mathbb{R}^4$ $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

S = all v in $\mathbb{R}^4$ with $v_1 + v_2 + v_3 + v_4 = 0$

yes this forms a subspace since

$$v^1 + v^2 = \begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \end{bmatrix} \Rightarrow v_a + v_b + v_c + v_d = 0$$

The dimension here is 3

What is the null space $Av = 0$?

= null space of $A = [1 \; 1 \; 1 \; 1]$ rank 1 matrix

$\dim N(A) = n - r$

$= 4 - 1 = 3$

basis of ~~the null sp~~ S

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \checkmark$$

column space of A

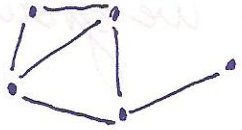
$C(A)$ is $\mathbb{R}^1$

what about left null space

$N(A^T) = \{0\} \rightarrow \overset{\dim}{m - r} = 1 - 1 = 0$

Small World graphs

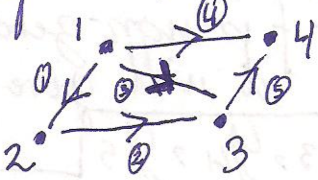
What is a graph in linear algebra = {nodes, edges}



Q - with a few shorcuts in the graph your distances between nodes drop dramatically?

Lecture 12 : Applications of Linear Algebra

Graph : Nodes, Edges



$n = 4$ nodes

$m = 5$ edges

Incidence matrix

$$A = \begin{array}{c} \text{node} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$

$$A = \left[\begin{array}{cccc} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \text{edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{loop}$$

Loops correspond to linearly dependent rows

$$Ax=0 \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \qquad \begin{aligned} -x_1+x_2&=0 \\ -x_2+x_3&=0 \\ -x_1+x_3&=0 \\ -x_1+x_4&=0 \\ -x_3+x_4&=0 \end{aligned}$$

lets give it this meaning $x = x_1, x_2, x_3, x_4$

$\boxed{e=Ax}$ potential at nodes A ↓

$x_2 - x_1$ e.tc are the potential diff *

What is in the null space $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

What is the basis of the null space of A is also $c\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ dim 1

In application it means if x is in the null-space nothing will move

→What if we ground node 4 i.e. $x_4 = 0$?

The <u>rank of A is 3</u> because only have 3 free variable and typically we ground the 4th node

<u>Null space of $A^T \rightarrow A^Ty=0$</u>

$$\underset{\substack{\downarrow \\ n\times m \\ 4\times 5}}{A^T} y = \underbrace{\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{f} \qquad \text{dim} = m - r = 5-3=2$$

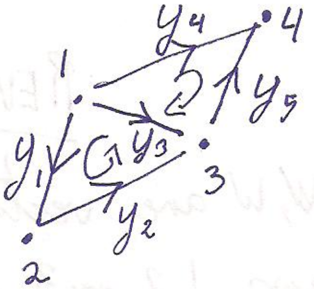
← f (if f is non-zero then it will give current sources)

* potential diff $\xrightarrow[\text{ohm's law}]{C}$ currents y_1, y_2, y_3, y_4, y_5 on edges

$\boxed{y=Ce}$

so $A^Ty=0$ Kirchoff's current law KCL

$A^T y \rightarrow$ ① $-y_1 - y_3 - y_4 = 0$



first eq. in node 1
(currents going out)

② $y_1 - y_2 = 0$ (for node 2)

③ $y_2 + y_3 - y_5 = 0$ (for node 3)

④ $y_4 + y_5 = 0$ (for node 4)

using elimination we can get null-space i.e a solution for the currents

basis for $N(A^T) \rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

current around loop 1

current around loop 2

so what about the whole loop $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ this is already in the basis

Pivot cols of $A^T \rightarrow$ col 1, 2, 4

they give the edges which don't have a loop
⇒ the rows are independent

Tree is a graph with no loops

$\dim N(A^T) = m - n$

$\boxed{\# \text{loops} = \# \text{edges} - (\# \text{nodes} - 1)}$
(rank = $n - 1$)

$\# \text{nodes} - \# \text{edges} + \# \text{loops} = 1$

Euler's formula

$\boxed{\boxed{A^T C A x = f}}$

Lecture 13

REVIEW

Q - Suppose V, W, W are vectors in $\mathbb{R}^7$ (non-zero)
subspace spans 1, 2, or 3 dimensions by V, V, W

Q - 5x3 U matrix (echelon form) and rank $r = 3$

→ What is $N(U)$ = zero vector = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now given 10x3 matrix $\begin{bmatrix} U \\ 2U \end{bmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{bmatrix} U \\ 0 \end{bmatrix}$

Now what about $C = \begin{bmatrix} U & U \\ U & 0 \end{bmatrix} \rightarrow \begin{bmatrix} U & U \\ 0 & -U \end{bmatrix}$

Going for rref $\begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \rightarrow \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$ the rank is 6

dimension $N(C^T) \longrightarrow \underset{10\times 6}{C} = 10 - 6 = 4$

Q = $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

dim row ~~of~~ space of A → since A is a 3x3 matrix

Since the solution vectors ~~are indeper~~ .....

What is the rank // the dim $N(A) = 2$

→ What is the matrix A?

$$A = \left[\begin{array}{c|c|c} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

because it is in the null space

$Ax = b$ can be solved for
→ when b has the form $b = c\begin{bmatrix}1\\2\\1\end{bmatrix}$

Q- If the $N(A) = \{0\}$ and A is square, what is the $N(A^T)$ would also be $\{0\}$

Q- Given a space of all 5x5 matrices, does the 5x5 invertible matrices form a subspace → No because I dont know if they add n give an invertible matrix

Q- $B^2 = 0$ does that imply $\Rightarrow B = 0$? False

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

Q- System of n equations and n unknowns, $n \times n$ matrix independent cols ... $Ax = b$ always solvable? Yes cause rank n

Q- $B = \begin{bmatrix}1&1&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&-1&2\\0&1&1&-1\\0&0&0&0\end{bmatrix}$ by not multiplying them

what is the basis for $N(B)$ — it will in $\mathbb{R}^4$

Since B is invertible, rank 3 ~~N(B) would be~~

$N(CD) = N(D)$ if C is invertible, hence basis for $N(B) = \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$

complete solution $Bx = \begin{bmatrix}1\\0\\1\end{bmatrix}$

$$x_{particular} + x_{null} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c\begin{bmatrix}\\ \\ \\ \end{bmatrix} + d\begin{bmatrix}\\ \\ \\ \end{bmatrix}$$

There are also other particular sols.

Q- If $m \times m$, the row space = col. space? False

but in case of symmetric matrices its true

Q- If A and B has the same 4 subspaces, then $A = cB$? If A is invertible, $N(A) = \{0\}$, $N(A^T) = \{0\}$ for a square matrix, hence it is false

many matrices have this

Q- If A, and exchange two rows, which subspaces stay same?

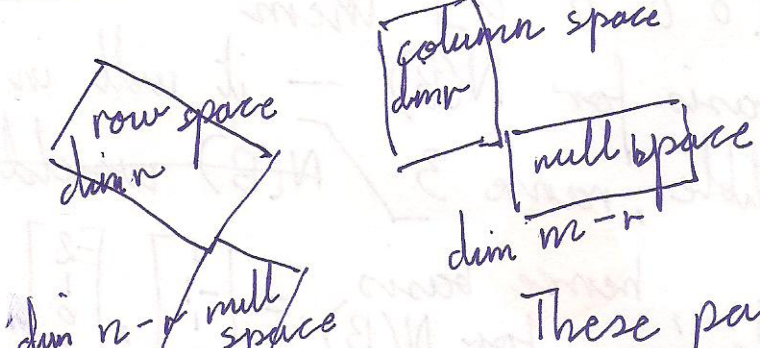
row space and null space would remain the same. but not col. space

Q- Why can't $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can't be in the null-space of a matrix & the row-space? or be in the row of A?

if $Av = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Now why cant $A = \begin{bmatrix} 1 & 2 & 3 \\ - & - & - \\ - & - & - \end{bmatrix}$? ofcourse!

$\therefore$ $N(A) \cap$ row space of $A = \{0\}$

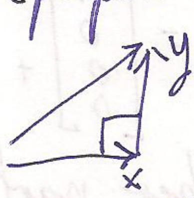
Lecture 14



These pairs are orthogonal subspaces

What are two orthogonal (perpendicular) vectors?

how can I tell two perpendicular vectors



$x^T y = 0$

$\|x\|^2+\|y\|^2=\|x+y\|^2$ only if $x^Ty=0$

Note $\|x\|^2 = x^Tx$, $x=\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}, y=\begin{bmatrix}2\\-1\\0\end{bmatrix}}_{\text{orthogonal vectors}}$

$\|x\|^2=14$, $\|y\|^2=5$

$\|x+y\|^2=19$

so check $x^Tx+y^Ty=(x+y)^T(x+y)$

$= x^Tx+y^Ty+x^Ty+y^Tx$

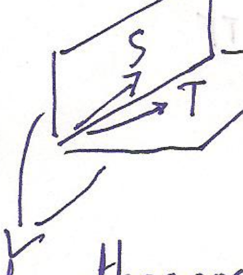
$\Rightarrow \quad 0 = 2x^Ty$

Test $\quad 0 = x^Ty$ ✓

→ The zero vector is orthogonal to everybody

Q. What are two orthogonal subspaces?

Subspace S is orthogonal to subspace T



It means: every vector in S is orthogonal to every vector in T

* If the subspaces intersect each other they cannot be orthogonal (except at origin)

not orthogonal (look at the intersection)

→ row space is orthogonal to the null space ... why?

$$Ax=0 \quad \begin{bmatrix}\text{row 1 of A}\\ \text{row 2} \ldots\\ \vdots\\ \text{row m of A}\end{bmatrix}\begin{bmatrix}\\ x\\ \\ \end{bmatrix}=\begin{bmatrix}0\\0\\0\\ \vdots\\0\end{bmatrix}$$

row1 . x = 0 , ... row m . x = 0 ∴

∴ By definition of the null space, row space & nullspace needs to orthogonal

What else is in the row space of A? the combination of the rows

$(c_1 \,\text{row}\,1 + c_2 \,\text{row}\,2 \dots c_m \,\text{row}\,m)\, x = 0$

$c_1 \,\text{row}\,1 . x + c_2 \,\text{row}\,2 . x + \dots c_m \,\text{row}\,m . x = 0$

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$ dim $r = 1$
dim $N(A) = 2$

$N(A)$ is a plane perpendicular to $[1\ 2\ 5]$

null space and row space are orthogonal <u>complements</u> in $\mathbb{R}^n$

Nullspace contains all vectors $\perp$ row space $(n - r + r = n)$

Coming up: solve $Ax = b$ when there is no solution $(m > n)$

Jumping ahead since there is error in b

$\underline{\underline{A^TA}} \rightarrow$ (1) its a square matrix

(2) its symmetric $\rightarrow (A^TA)^T = A^TA^{TT}$

So for $Ax = b \rightarrow A^TAx = A^Tb$

(3) When is A^TA invertible?

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} \rightarrow \text{solvable if } b \text{ is in the column space of } A$$

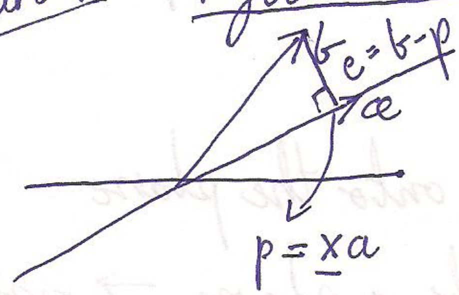
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 8 & 30 \end{bmatrix} \rightarrow \text{It is invertable}$$

But if A was rank deficient, then A^TA wouldn't be invertible

Rank of A^TA = rank of A

A^TA is invertible iff A has independent columns

Lecture 15 Projections in 2D



b projects onto line a, using an orthogonal line from a at point p. where p is a scalar multiple $p = xa$

$$a^T(b - xa) = 0$$

$$xa^Ta = a^Tb$$

$$x = \frac{a^Tb}{a^Ta}, \quad \underline{p = ax}$$

projection

So $p = a\frac{a^Tb}{a^Ta}$

* If b goes twice as far, the projection will double too

* If a goes twice as far, the projection will remain same

$$\text{proj } p = \underline{P}b = \boxed{\frac{aa^T}{a^Ta}}b$$

projection matrix (aa^T is a whole matrix)

Facts about P (projection matrix)

* Column space $C(P)$ = line through a

* Rank$(P) = 1$

* P is symmetric $\rightarrow P^T = P$

* Projection multiples times gives the same matrix $P^n = P$ for $n \in \mathbb{Z}, n \geq 1$

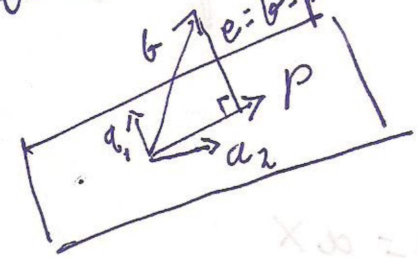
Why project?

Because $Ax=b$ may have no solution → So the solution is to get the closest b

We want a b which is in the col. space of A, but it is not

→ solve $A\hat{x}=p$ instead proj. of b onto col. space

Projections in 3D



projection of b onto the plane

* How to describe a plane → just give 2 basis vector (a_1, a_2).

i.e. we need an A which has a_1 and a_2 in the column space $A=\begin{bmatrix} a_1 & a_2 \end{bmatrix}$

e is perpendicular to the plane

$$\begin{cases} p = \hat{x}_1 a_1 + \hat{x}_2 a_2 \\ p = \underset{\text{the projection matrix}}{A}\hat{x} \end{cases}$$

line in the plane

Find $\hat{x}$?

So we have: $a_1^T(b-A\hat{x})=0$, $a_2^T(b-A\hat{x})=0$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}\underline{(b-A\hat{x})}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

e is in $N(A^T)$ which is orthogonal to column space $C(A)$

$$A^T(b-A\hat{x})=0$$

$$\underline{A^TA\hat{x}=A^Tb}$$

$$\underline{\hat{x}=(A^TA)^{-1}A^Tb} \longrightarrow \text{from } p=A\hat{x}$$

$$p=\underline{A(A^TA)^{-1}A^T}b$$

P the projection matrix

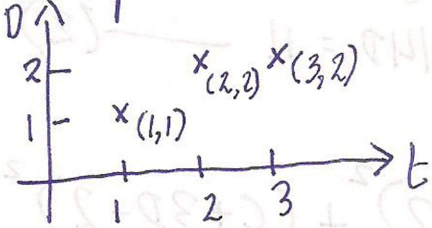
Aah!! the moore penrose

Notice $P=I$, if A is the whole space in $\mathbb{R}^n$

Checking the fact's for P

- $P^T = A^{TT}(A^TA)^{-1T}A^T = P$
- Rank is of course 1
- $P^2 = A(A^TA)^{-1}\cancel{A^TA(A^TA)^{-1}}A^T$

Given 3 points (our data)



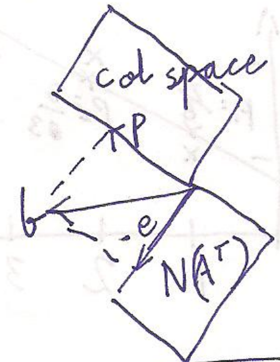
Problem is to fit this to a line

$b = C + Dt \Longrightarrow$

$$\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned} \Rightarrow \overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

Lecture 16 $p = Pb$

- If b is in column space $Pb = b$
- If $b \perp$ column space $Pb = 0$



$$p + e = b$$

$$e = (I - P)b$$

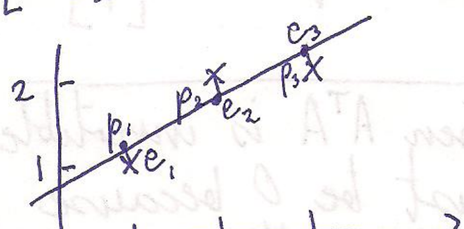
projc onto $\perp$ space

Find the best straight line $b = C + Dt$

What is the best solution? Minimize the sum of squared errors we make on each eq. $Ax - b = e$

$$\underline{\min \|Ax - b\|^2 = \min \|e\|^2}$$

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \longrightarrow \min e_1^2 + e_2^2 + e_3^2 \longleftarrow \text{(this is linear regression)}$$



$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ is in the column space

↑ replaces b (see the col. space, $N(A^T)$ picture above)

Find $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$ & p

$$A^TA\hat{x} = A^Tb$$

$$\underset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} = \underset{A^TA}{\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}} \rightarrow \text{symmetric and invertable}$$

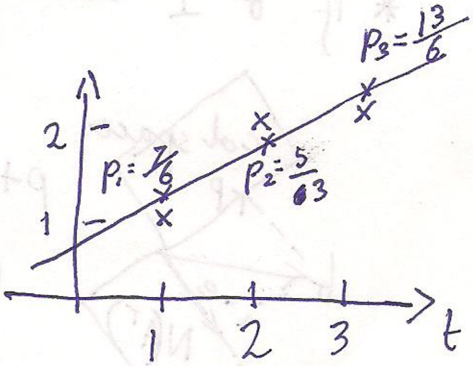
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right] \quad \begin{array}{l} 3C + 6D = 5 \text{ —— (1)} \\ 6C + 14D = 11 \text{ —— (2)} \end{array}$$

$$e_1^2 + e_2^2 + e_3^2 = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$$

taking partial derivative $\frac{\partial}{\partial D}$ and $\frac{\partial}{\partial C}$ will give equation (1) & (2)

Solving (1) & (2) by back-substitution $\rightarrow$ $2D = 1$, $3C = 2$

So we have $b = \frac{2}{3} + \frac{1}{2}t$



$$\boxed{\begin{aligned} e_1 &= \tfrac{1}{6} \\ e_2 &= -\tfrac{1}{3} \\ e_3 &= \tfrac{1}{6} \end{aligned}}$$

$$b = \overset{\text{proj vector}}{p} + \overset{\text{error vector}}{e}$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} 1/6 \\ -1/3 \\ 1/6 \end{bmatrix}$$

p should be $\perp$ e $\Rightarrow p \cdot e = 0$

* Lets take any other vector in the col. space $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
Now this is also $\perp$ to e

If A has independent columns, then A^TA is invertible

Prove Suppose $A^TAx = 0$, now x must be 0 because there should be nothing else in the null space of A^TA

$$\rightarrow x^TA^TAx = 0 \Rightarrow (Ax)^T(Ax) = 0 \Rightarrow Ax = 0$$

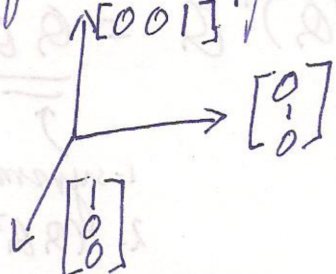
If A has independent cols and if $Ax = 0$, $x = 0$!!

* Columns are definitely independent if they are <u>perpendicular unit vectors</u>

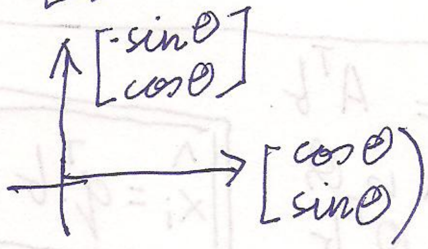
orthogonal

||

orthonormal vectors



another orthonormal vector



Lecture17

Orthogonal vectors now referred to as $q_1, q_2 \cdots q_n$ orthonormal vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\underset{\uparrow}{Q} = \begin{bmatrix} q_1 \cdots q_n \end{bmatrix} \qquad Q^T Q = I$$

orthogonal (normal) matrix / Usually called so when Q is square - then not only $Q^TQ = I$ it also tells us $Q^T = Q^{-1}$

Examples perm $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $QQ^T = I$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}, \quad Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \leftarrow \text{adamar matrix}$$

Lets say Q has orthonormal columns

Project onto its column space

if Q is square

$$P = Q(Q^TQ)^{-1}Q^T = \underline{\underline{QQ^T}} \{ = I \}$$

1- symmetric

2. $(QQ^T)(QQ^T) = I$

$A^TA\hat{x} = A^Tb$

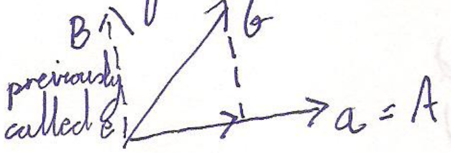
Now A is Q

$\underline{I\hat{x} = Q^Tb}$

$$\boxed{\hat{x}_i = q_i^Tb}$$

How to make independent vectors as orthogonal

<u>Gram-Schmidt</u> independent vectors a, b → orthogonal vectors A, B → orthonormal



$q_1 = \frac{A}{\|A\|}$, $q_2 = \frac{B}{\|B\|}$

Set a = A

Now we need to subtract the projection of b onto a from b → $\boxed{B = b - \frac{A^Tb}{A^TA}A}$

To check the answer $A^TB = A^Tb - \cancel{A^T}\frac{A^Tb}{\cancel{A^TA}}\cancel{A} = 0$

How to get a third orthogonal vector C

$$C = c - \underbrace{\frac{A^Tc}{A^TA}}_{\text{projection in A}}A - \underbrace{\frac{B^T\cancel{b}c}{B^TB}}_{\text{projection in B}}B$$

Calculating Gram Schmidt doesn't change the column space

example $a = [1\ 1\ 1]^T$, $b = [1\ 0\ 2]^T$

$$B = \begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{[1\,1\,1]\begin{bmatrix}1\\0\\2\end{bmatrix}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$Q = \begin{bmatrix}1/\sqrt{3} & 0\\ 1/\sqrt{3} & -\frac{1}{\sqrt{2}}\\ 1/3 & \frac{1}{\sqrt{2}}\end{bmatrix}^3$$

$A = QR = [a_1\ a_2] = \begin{bmatrix} q_1 q_2 \end{bmatrix} \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ a_1^T q_2 & a_2^T q_2 \end{bmatrix} \cancel{= \begin{bmatrix} a_1^T q_1 & 0 \\ 0 & a_2^T q_2 \end{bmatrix}}$

$\cancel{[q_1 q_2]}$

factorization $\rightarrow A = \underline{QQ^T A}$

$R = Q^T A$

Lecture 18

Determinant is a no. associated with a sq. matrix

det $A = |A|$

Singular (non-invertible) matrix det $A = 0$

Properties of determinant

1- $|I| = 1$

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

2- Exchanging rows of A reverse sign of det

det $P = \begin{cases} 1 & \text{even exchanges} \\ -1 & \text{odd exchanges} \end{cases}$

$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

3a- $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3b- $\begin{vmatrix} a+a' & b+b' \\ \vdots & \vdots \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

det ~~it~~ behaves like a linear combination if you change a single row

Sub-properties

4- 2 equal rows $\rightarrow |A| = 0$

this can be shown by ② exchanging those rows

5- Subtract $l \times$ row 1 from row K (elimination) $\rightarrow$ this doesn't change the det.

$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \overset{ⓑ3b}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$

6- Row of zeros $\rightarrow |A| = 0$

$\begin{vmatrix} 0.0 & 0.0 \\ c & d \end{vmatrix} = 0 \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$

7- $U = \begin{bmatrix} d_1 & x & x & x \\ 0 & d_2 & x & x \\ \vdots & & \ddots & \\ 0 & 0 & 0 & d_n \end{bmatrix}$ $|U| = d_1 \times d_2 \cdots \times d_n$

Softwares use this fact with eliminat to comute det

making U in RREF (rule 5) 3a to

8- det A = 0, when A is singular
From rule 7, we know that we will have a row of zeros which will cause the det A = 0

$$\det\begin{bmatrix} \boxed{a} & b \\ c & d \end{bmatrix} \text{ using elimination } \begin{matrix} a & b \\ 0 & d-\frac{c}{a}b \end{matrix}$$

using rule 7 we know $\rightarrow |A| = ad - cb$

9- $|AB| = |A| \times |B|$

so for $|A^{-1}| = \frac{1}{|A|}$

a- $\det A^2 = (\det A)^2$

b- $\det 2A = 2^n \det A$ (when $A_{n\times n}$) rule 3a

10- $\det A^t = \det A$

$|A^T| = |A|$

$|U^T L^T| = |LU|$

$|U^T||L^T| = |L||U|$ ✓

Lecture 19 Formula for determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}^{\text{singular}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}} \leftarrow \text{singular}$$

steps following linearity (rule 3b)

$$= ad \underbrace{-}_{} cb$$

(- because we did a flip)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} +$$

(survivors only have 1 entry each in a row & col

The Formula

Since we are always looking for permutation of n rows, the resulting terms in the formula are n!

$$\det A = \sum_{n!\ \text{terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \dots a_{n\omega}$$

$(\alpha, \beta, \gamma, \dots, \omega)$ = Permutation of $(1, 2, \dots n)$

Example : $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \text{Perm}(4,3,2,1) \pm \text{Perm}(\overset{3}{\cancel{4}},2,1,4)$

$= +1 - 1 = \underline{0}$ singular matrix

Cofactors 3x3 in parentheses

$$\begin{aligned} \det = {} & a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ & - \cancel{+} a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ & + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

the cofactor is the det of the smaller matrix

$$\underset{C_{ij}}{\text{cofactor of } a_{ij}} = \pm\det\begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \ \& \text{ col } j \\ \text{erased} \end{pmatrix}$$

also called minor $i+j$ is even $i+j$ is odd

once the sign is built in it

Cofactor formula : $\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \dots a_{1n}C_{1n}}$

with signs incorporated

Example

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = |A_4| = 1.C_{11} + 1.C_{12}$$

$$= 1.(\cancel{+}(0-1+0)) + 1.(-(0\cancel{x}-0))$$

$$= -1$$

Lecture 20

Applications of the det

Appl 1: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{|A|} C^T}$$

C^T ← matrix with cofactor entries

involves n-1 entries

$|A|$ ↑ involves ~~n entries~~ sums of product of n entries

Checking $AC^T = (\det A) I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & \vdots \\ \vdots & & \ddots & \\ C_{1n} & \cdots & & C_{nn} \end{bmatrix} = \begin{bmatrix} |A| & & 0 \\ & |A| & \\ 0 & & |A| \end{bmatrix}$$

↑ off-diagonals become 0 because we are taking the det. of a matrix which is singular

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ ←

$A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix} = ab - ab = 0$ linearly dependent cols

---

Appl 2: $Ax = b$

$$x = A^{-1}b = \frac{1}{|A|} C^T b$$

CRAMER'S RULE

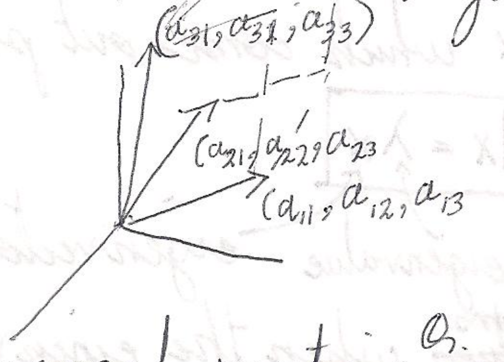
$$x_1 = \frac{\det B_1}{|A|} \qquad x_2 = \frac{\det B_2}{|A|} \qquad x_j = \frac{|B_j|}{|A|}$$

where $B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \\ \text{cols} \\ \text{of } A \end{matrix} \end{bmatrix}$ = A with col. 1 replaced by b

so it is about calculating n+1 determinants

Appl. 3

$|\text{Det } A|$ is the volume of a box where the edges of the box are given by the rows



The sign tells us if its a right-handed or left-handed box

* An orthonomal matrix Q is a unit cube rotated around. So why is $|Q| = \pm 1$?

Since $Q^TQ = I$

$|Q^TQ| = |I|$

$|Q^T||Q| = |I|$

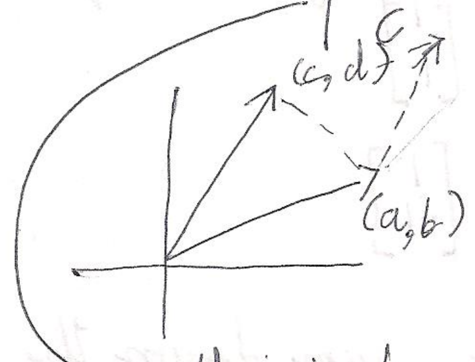
$|Q|^2 = 1$

$|Q| = \pm 1$

* Lets say we double a side, the volume doubles. This was rule 3a of determinants

* Infact see all 3 rules of determinants satisfy volume calculation

rule3b: $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

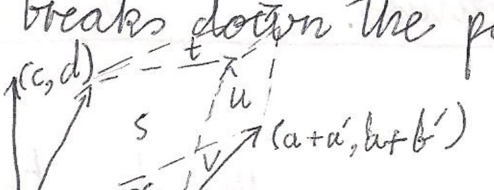


area = ad - cb

(rather than the dirty formula of the parallelogram)

area of triangle $= \frac{1}{2}(ad - bc)$

→ this is true because the decomposition just breaks down the parallelogram into two with one same side



$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Lecture 21 Eigenvalues/Eigenvectors

We are interested in Ax which come out parallel to x.

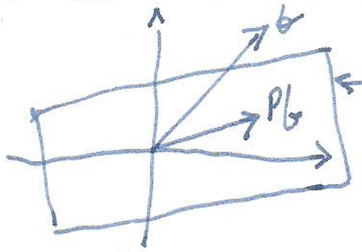
Ax Eigenvectors parallel to x

$$\boxed{Ax = \underset{\text{eigenvalue}}{\lambda}\ \underset{\text{eigenvector}}{x}}$$

* What are the eigen~~values~~ vectors when the eigenvalue is in the nullspace of A. $Ax=0$

* If A is singular, $\lambda=0$ is an eigenvalue

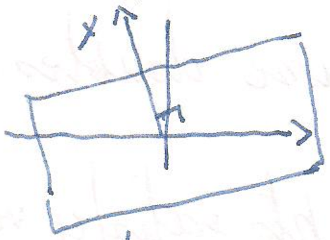
→ What are the x's and λ's for a projection matrix?



← So b is not an eigenvector for A.

but x in plane: $Px = x$ → here the $\lambda = 1$

so $x \perp$ plane: $Px = 0x$ → here the $\lambda=0$



Example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ← permutation matrix

What could be the eigenvector for $\lambda = 1$?

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

And for $\lambda = -1$? $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

* Sum of eigenvalues $\sum_i \lambda_i = a_{11} + a_{12} + \dots a_{nn}$ sum down the diagonal.

* nxn matrixes have n eigenvalues.

How to solve $Ax = \lambda x$?

rewrite → $(A - \lambda I)x = 0$

$|A-\lambda \underline{I}|=0$ find λ first, after that x found by finding the nullspace. ㉕

Example $A=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} \rightarrow |A-\lambda \underline{I}| = \begin{vmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix}$

$= (3-\lambda)^2 - 1$

$0 = \underline{8} - \underline{6\lambda} + \lambda^2 = (\lambda-4)(\lambda-2)$

determinant in 2x2 case

trace? in 2x2 case

trace = $\overset{3}{\cancel{0}}+\overset{3}{\cancel{0}} = 6 = \lambda_1+\lambda_2$

det. = $8 = \lambda_1\lambda_2 = 4\times 2$

So let's find x for $\lambda = 4$..

$A - 4\lambda = \underbrace{\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}} \rightarrow x_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$

remember matrix should be singular

Now for $\lambda = 2$

$A - 2\lambda = \begin{bmatrix}1 & +1\\ 1 & 1\end{bmatrix} \quad x_2 = \begin{bmatrix}-1\\ 1\end{bmatrix}$

* What happens when we change something about A to the eigenvectors & eigenvalues?

$\rightarrow Ax = \lambda x \longrightarrow (A + a\underline{I})x = \lambda x + ax = (\lambda + a)x$

$\rightarrow$ If $Ax = \lambda x$, B has eigenvalues α. $\rightarrow$ but $(A+B)x \neq (\lambda+\alpha)x$ because we have no reason to believe x is also an eigenvector of B

Example $Q = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \rightarrow |Q - \lambda\underline{I}| = \begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix} = \lambda^2 + 1 = 0$

90° rotation

If we take symmetric matrices or close to symmetric, they

eigenvalues can have $\rightarrow \lambda_1 = i$, $\lambda_2 = -i$

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \rightarrow$ If a matrix is triangular then the eigen values are on the diagonal

$$|A - \lambda I| = (3-\lambda)^2 \rightarrow \lambda_1 = 3, \; \lambda_2 = 3$$

$$(A - \lambda I) = x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underset{\text{in the nullspace}}{\begin{bmatrix} x \end{bmatrix}} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\underbrace{x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \dots, x_k = \begin{bmatrix} a \\ 0 \end{bmatrix}}$$

but this cant be true because they are not independent $\rightarrow x_2 = \underline{\text{No}}$ independent x_2

Lecture 22

Given an eigenvector matrix S

Suppose we have n linearly indep. eigenvectors of A

Put them in columns of S

$$AS = A\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \overset{\lambda_1}{\cancel{A}} x_1 & \overset{\lambda_2}{\cancel{A}} x_2 & \dots & \overset{\lambda_n}{\cancel{A}} x_n \end{bmatrix} = \underset{S}{\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}} \underset{\Lambda}{\overset{\text{eigenvalue matrix}}{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}}}$$

So $AS = S\Lambda$

$\boxed{S^{-1}AS = \Lambda}$ or $A = S\Lambda S^{-1} \leftarrow$ <u>factorization</u>

S will ~~alw~~ be invertible if they have n independent eigenvectors

* What about eigen values/vectors for A^2

$Ax = \lambda x$ | or $A = SAS^{-1}$

$\rightarrow A^2 x = \lambda A x$ | $A^2 = SAS^{-1}SAS^{-1}$

$A^2 x = \lambda^2 x$ same λ^2 | $= SA^2S^{-1}$ same

* What about A^K

$$A^K = S\Lambda^K S^{-1}$$

because $= S\Lambda S^{-1} S\Lambda S\Lambda \ldots\ldots A^{(k)} S^{-1}$

$= S\Lambda^K S^{-1}$

Theorem $A^K \longrightarrow 0$ as $K \to \infty$

since $A^K = S\Lambda^K S^{-1}$

if all $|\lambda_i| < 1 \quad \forall i$

A is sure to have n independent eigenvectors (and be diagnolizable)

if all the λ's are different (no repeated λ's)

For repeated eigenvalues // I may or may not have n independent eigenvectors

if $A = I$ then all eigenvalues are 1, but there is no shortage of eigenvectors

Example $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} \quad \lambda = 2, 2$

so $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but no other independent x

---

Start with given vector u_0

eg. $u_{K+1} = Au_K$, $u_1 = Au_0$, $u_2 = Au_1 = A^2 u_0$ $\boxed{u_K = A^K u_0}$

To solve this : write $u_0 = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

x_i need to be normalized

$$Au_0 = c_1 \lambda_1 \underline{x_1} + c_2 \lambda_2 x_2 + \ldots\ldots + c_n \lambda_n x_n$$

so $A^K u_0 = c_1 \lambda_1^K x_1 + c_2 \lambda_2^K x_2 + \ldots\ldots + c_n \lambda_n^K x_n$

$u = \Lambda^{k} S c$

Example: Fibonacci numbers
$0, 1, 1, 2, 3, 5, 8, 13, \ldots\ldots F_{100} = ?$

So $F_{K+2} = F_{K+1} + F_K$

so taking $u_K = \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix} \longrightarrow u_{K+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix}$

and $F_{K+1} = F_{K+1}$

* For symmetric matrices the eigenvalues will be real

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

$\lambda_1 \approx 1.618$
$\lambda_2 \approx -0.618$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = 0 \qquad \frac{1 \pm \sqrt{1+4}}{2} \Rightarrow \lambda_1 = \frac{1}{2}(1+\sqrt{5})$$

$$\lambda_2 = \frac{1}{2}(1-\sqrt{5})$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

When things are evolving in time by a 1st order system, starting from an orig. u_0, ~~th~~ ($u_{K+1} = Au_K$) the Key is to find the eigenvectors of A; those eigenvectors will already tell you whats happening i.e. how is the solution evolving. The take u_0 and write it as a combination of eigenvectors and follow each eigen vector separately

Lecture 23 differential eq. $\frac{du}{dt} = Au$

Example $\frac{du_1}{dt} = -u_1 + 2u_2$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

Initial condition $du_2 \quad = u_1 - 2u_2$

$\lambda_1 = 0 \quad , \quad \lambda_2 = -3$

because matrix is singular ↑   ↑ from trace

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftarrow Ax_1 = 0x_2 \quad , \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \leftarrow Ax_2 = -3x_2$

Solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

check: $\dfrac{du}{dt} = Au$ Plug in $e^{\lambda_1 t} x_1 \rightarrow \lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$

so $\lambda_1 x_1 = Ax_1$

$$\rightarrow u(t) = c_1 x_1 + c_2 e^{-3t} x_2$$

$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Given $u(0) \rightarrow c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$c_1 = \frac{1}{3} , \quad c_2 = \frac{1}{3}$$

$$\therefore u(t) = \frac{1}{3} x_1 + \frac{1}{3} e^{-3t} x_2$$

$$= \underbrace{\frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\text{steady state}} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

① Stability $u(t) \rightarrow 0$ we need $e^{\lambda t} \rightarrow 0$ $Re(\lambda) < 0$, we can ignore the imaginary part

② Steady: $\lambda_1 = 0$ and other $Re(\lambda) < 0$

③ Blowup if any $Re(\lambda) > 0$

2×2 stability $Re\,\lambda_1 < 0$, $Re\,\lambda_2 < 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\left.\begin{matrix} \text{trace } a+d = \lambda_1 + \lambda_2 < 0 \\ \det = \lambda_1 \lambda_2 > 0 \end{matrix}\right\}$ for stability of 2×2 matrix

---

for the above example $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for initial condition

[illegible] in terms of S & Λ

Sol: $u = Sv$ given $\frac{du}{dt} = Au$

$$v(t) = e^{\Lambda t} v(0)$$

$$S\frac{dv}{dt} = ASv$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0)$$

$$\frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

So $e^{At} = S e^{\Lambda t} S^{-1}$

How to define matrix exponential

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \ldots\ldots + \frac{(At)^n}{n!} + \ldots\ldots$$

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$ from Taylor series $$\frac{1}{1-x} = \sum_0^\infty x^n$$

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \ldots\ldots$$

$$= I + \underbrace{S\Lambda S^{-1}}_{A} t + \frac{S\Lambda S^{-1} t^2}{2} + \ldots\ldots, = \boxed{S e^{\Lambda t} S^{-1} = e^{At}}$$

$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ So what is the exponential of a diagonal matrix $\rightarrow e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$

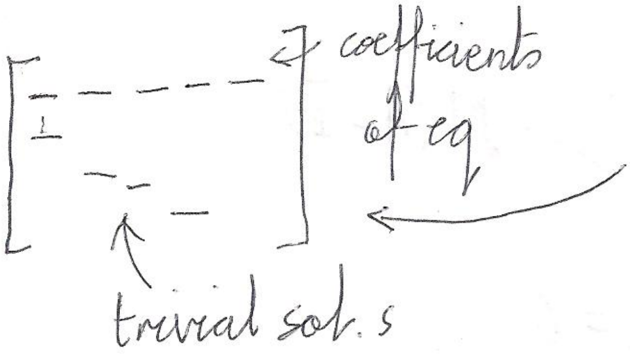
$\boxed{e^{At} \text{ will go to zero } \text{~~at~~ iff } Re(\lambda_i) < 0}$

Example: $y'' + by' + ky = 0$ How do I turn a second order equation to a first-order system

~~replate~~ replace it by $u = \begin{bmatrix} y' \\ y \end{bmatrix}$ $u' = \begin{bmatrix} y'' \\ y' \end{bmatrix}$

so $$\begin{bmatrix} -b & -K \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix} = \begin{bmatrix} y'' \\ y' \end{bmatrix} = u'$$

In short



5th order to 5x5 1st order

Lecture 24

Markov matrices

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

Appl of eig values

Property ① All entries ≥ 0
Property ② All cols add to 1

1. $\lambda = 1$ is an eigenvalue
2. All other $|\lambda_i| < 1$

$$u_K = A^K u_0 = c_1 \underset{\underset{1}{\parallel}}{\lambda_1^K} x_1 + c_2 \underset{\underset{<1}{\parallel}}{\lambda_2^K} x_2 + \ldots\ldots \rightarrow c_1 x_1 \geq 0$$

as $K \to \infty$, $\lambda_2 = 0$

steady state

$$A - \underset{\underset{\lambda_1}{\uparrow}}{1} I = \underbrace{\begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}}_{\text{all cols add to } 0}$$

→ $A - I$ is singular

this is so because if ~~I~~ I add all the rows will give 0, i.e the rows are dependent because this $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is in $N(A^T)$

Also x_1 will be in $N(A)$

So what is known about the eigenvalue of A and A^T?

they are the same!

Proof: $\det(A - \lambda I) = 0$

$$\Downarrow$$

$\det(A^T - \lambda^T I) = 0$

$\det(A^T - \lambda I) = 0$

$$\cancel{\begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix} \begin{bmatrix} .4 \\ .33 \end{bmatrix}}$$

$\mathcal{N}(A-I)$

$$\begin{bmatrix} -.9 & .01 & .3 \\ .2 & -0.1 & .3 \\ .7 & 0 & -.6 \end{bmatrix} \begin{bmatrix} .6 \\ 33 \\ .7 \end{bmatrix} = 0$$

E.g. Movement of ppl between cali and mass

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=K+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=K}$$

Shows movement of ppl → like 20% ppl from mass move to cali

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} \rightarrow \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=1} = \begin{bmatrix} 200 \\ 800 \end{bmatrix} \rightarrow \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=2} = \begin{bmatrix} 340 \\ 640 \end{bmatrix}$$

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \quad \lambda_1 = 1, \lambda_2 = .7$$

$$\underbrace{\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}}_{A-I} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x_1} = 0$$

$$\underbrace{\begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix}}_{A-.7I} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x_2} = 0$$

$$u_K = c_1 1^K \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^K \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \underset{\frac{1000}{3}}{c_1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \underset{\frac{2000}{3}}{c_2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_\infty = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$u_{100} = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} (.7)^{100} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Projection with orthonormal basis

App 2 of eig

$q_1, \dots q_n \leftarrow$ orthonormal basis

Any

$x q + x q + \dots + x_n q_n$

So taking

$$q_1^T v = x_1 q_1^T q_1 + 0 \ldots\ldots + 0 \rightarrow \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v$$

$$Qx = v$$

$$x = Q^{-1} v = Q^T v$$

$$x_1 = q_1^T v$$

Fourier series

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \ldots\ldots$$

as compared to above this is infinite series

basis is $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\ldots$

what does it mean for functions to be orthogonal?

$$\langle f, g \rangle = f^T \cdot g = \int f(x) g(x) dx$$

in the above case $\rightarrow \int_0^{2\pi} f(x) g(x) dx$

$$= \int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} (\sin x)^2 \Big|_0^{2\pi} = 0$$

So how do I get the 1st fourier coefficient $= a_1$

$$\cos(x) f(x) = a_1 \cos^2 x$$

$$\text{so} \quad a_1 \int_0^{2\pi} (\cos x)^2 dx = \int_0^{2\pi} f(x) \cos(x) \, dx$$

$$a_1 \pi = \int_0^{2\pi} f(x) \cos(x) dx$$

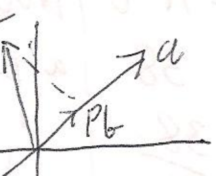
Lecture 24B

orthonormal matrix

$$Q = [q_1 \cdots q_n] \Rightarrow Q^T Q = I$$

Q About projection $A(A^TA)^{-1}A^T$

$$a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad P = \frac{aa^T}{a^T a} = \frac{1}{\;}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$



The rank of P is 1. And the column space of P is a ~~unles~~ itself.

~~The~~ Out of the three eigen value, two are zero because rank is 1

$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 1$

Eigenvector for λ_3. Its the eigenv that doesnt move

so $x_3 = a$

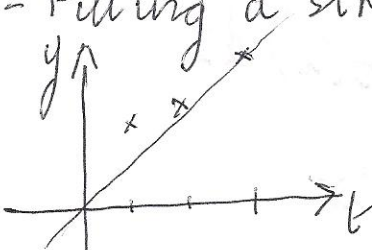
$u_{k+1} = Pu_k$

$u_0 = \begin{bmatrix}9\\9\\0\end{bmatrix},\ u_1 = Pu_0 = a\frac{a^T u_0}{a^T a} = a\frac{27}{9} = \overset{\hat{x}}{3}a = \begin{bmatrix}6\\3\\0\end{bmatrix}$

$u_k = P^k u_0 = Pu_0 = \begin{bmatrix}6\\3\\0\end{bmatrix}$

if P is not a projection, then to solve u_k, we would need decompose it as: $A^k u_0 = c_1 \underbrace{\lambda_1^k}_{=0} x_1 + c_2 \underbrace{\lambda_2^k}_{=0} x_2 + c_3 \underbrace{\lambda_3^k}_{=1} x_3$

Q- Fitting a straight line through the origin



$t = 1 \quad y = 4$

$= 2 \quad = 5$

$= 3 \quad = 8$

$y = Dt$

$\left.\begin{matrix}1D = 4\\ 2D = 5\\ 3D = 8\end{matrix}\right\}$ need to satisfy these eq.

$\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{A} \underbrace{D}_{x} = \underbrace{\begin{bmatrix}4\\5\\8\end{bmatrix}}_{b}$

Best D $\quad A^T A \hat{D} = A^T b$

$14\hat{D} = 38$

$\hat{D} = 38$

Projecting b onto col. space of A (line)

Now if we have 2 vectors in a plane $a_1 = \begin{bmatrix}1\\2\\3\end{bmatrix},\ a_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ – vector orthogonal to a_1 where $B \perp a_1$

$$\overset{a_2}{=} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{a_1^T \cancel{b} a_2}{a_1^T a_1} a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Q – 4x4 matrix, $\lambda_1, \lambda_2 \lambda_3, \lambda_4$ eigenvectors

→ ~~for a~~ if $\lambda_K \neq 0$, then the matrix will be invertible

→ $\det A^{-1} = \frac{1}{\lambda_1} \frac{1}{\lambda_2} \frac{1}{\lambda_3} \frac{1}{\lambda_4}$

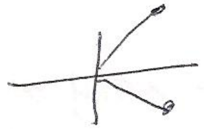
→ trace of $A+I = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4$

Q – $A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $D_n = \det A_n$

use cofactors $D_n = \_D_{n-1} + \_D_{n-2}$

$C_{11} = 1$, $C_{12} = 0$ $= D_{n-1} - D_{n-2}$

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$



eigen values

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{-3}}{2} \rightarrow \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2} \rightarrow e^{i\pi/3}, e^{-i\pi/3}$$

complex conjugates

So $\left(e^{i\pi/3}\right)^6$ brings us to the 0th line

Since $e^{2\pi i} = 1$

Q – $A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} = A_4^T \rightarrow A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ $P = A(A^TA)^{-1}A^T$

find eigenvalues & eigenvectors

$|A_3 - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \end{vmatrix} = -\lambda^3 + 5\lambda = 0$

Asks the projection matrix onto col. space of A_4

$|A_4| = +9$ so A_4 is invertible

the its colspace is R_4 that is the whole space

and then the projection matrix is simply the I

Lecture 25

Symmetric matrices $A = A^T$

① The eigenvalues are all REAL

② The eigenvectors are PERPENDICULAR (ORTHOGONAL)

* if the eigenvalues ~~are all~~ are repeated, then there is a plane of eigenvectors from which we can choose perpendicular ones

Usual case $A = S\Lambda S^{-1}$

② tells us that we can make orthonormal eigenvectors hense S can become Q

In the symmetric case ~~$A = Q\Lambda Q^{-1}$~~

$$\boxed{A = Q\Lambda Q^T}$$

→ So why only real eigenvalues

$$Ax = \lambda x \underset{\text{always}}{\Longrightarrow} \bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

conjugate $\overline{a+ib} = a-ib$

So ~~$A\bar{x} = \bar{\lambda}\bar{x}$~~ $\bar{x}^T A^T = \bar{x}^T\bar{\lambda} \rightarrow \bar{x}^T A = \bar{x}^T\bar{\lambda}$

↑ incase A is only real

or $\begin{cases} \bar{x}^T A\bar{x} = \bar{\lambda}\bar{x}^T\bar{x} \\ \bar{x}^T A x = \bar{x}^T\bar{\lambda} x \end{cases}$

but from $Ax = \lambda x \Rightarrow \bar{x}^T A x = \bar{x}^T\lambda x$

hence $\lambda = \bar{\lambda}$

$$\lambda\bar{x}^T x = \bar{\lambda}\left(\bar{x}^T x\right) = [\bar{x}_1, \bar{x}_2, \bar{x}_3 \cdots]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \bar{x}_3 x_3 \ldots$$

$(a-ib)(a+ib)$

$\underline{A = Q \Lambda Q^T}$ for $A = A^T$

$$\begin{bmatrix} q_1 & q_2 & \cdots \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \end{bmatrix} = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \ldots\ldots$$

but we know that $q_1 q_1^T$ is a projection matrix

* So every symmetric matrix is a combination of perp. projection matrices

* When I have symmetric matrices then we have eigenvalues that are real. Then would they be +ve or -ve

Maybe if we have 50x50 matrix, a quicker way to find this is to find the pivots, then the sign of pivots = signs of λ

\# +ve pivots = # +ve pivots

* for a symmetric matrix product of pivots = product of λs

Now wat is a <u>Positive Definite Matrix</u>

1- It is always symmetric

2- ∴ eigenvalues are real and <u>positive</u>

3- all pivots are <u>positive</u>

e.g. $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \longrightarrow$ pivots $5, \frac{11}{5}$

$$\lambda^2 - 8\lambda + 11 = 0$$

$$\lambda = 4 \pm \sqrt{5}$$

4- determinat is not only positive, but all subdeterminants are positive i.e. $|[5]|, \begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} > 0$

the top |x|

Lecture 26

Complex Vectors and Matrices

$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in C^n$ length $z^T z$ = not the right answer because now here $z_i^T z_i < 0$

e.g $(1+2i)(1+2i) = 1-4+4i$

what we want is $\bar{z}_i z_i = |z_i|^2$

$(1-2i)(1+2i) = 1+4$

hermitian

$z^H z$ so inner product $\bar{y}^T x = y^H x$

$\parallel$

$|z_1|^2 + \dots + |z_n|^2$

→ Symmetric matrices $A^T = A$ no good for complex A

so $A^T = A \longrightarrow \bar{A}^T = A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} = A^H \leftarrow$ these have real eigenvalues and orthonorm ~~perpen~~ eigenvectors

→ How do I check $\perp$? $\nwarrow$ orthonormal

$q_1, q_2 \dots q_n$

$\bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$

$Q^T Q = I \rightarrow$ for complex $Q^H \Rightarrow \bar{Q}^T Q = I$

Fourier Transform

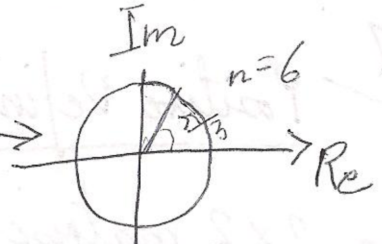
FFT reduces complexity from $n^2 \longrightarrow \frac{1}{2} n \log n$

$F_n = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 1 & W & W^2 & \cdots & W^{n-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(n-1)} \\ \vdots & & \vdots & & \vdots \\ & W^{n-1} & W^{2(n-1)} & & W^{(n-1)(n-1)} \end{bmatrix}$ $(F_n)_{ij} = W^{ij}$

where $i,j = 0 \dots n-1$

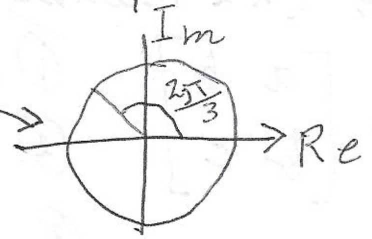
~~Its~~ W is a special number :

$W^n = 1 \rightarrow w = (e^{i2\pi/n = 1})$



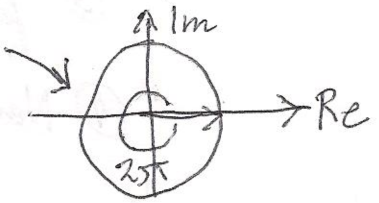
$\downarrow$

$W = \left(e^{i2\pi/n}\right)^2$



$\vdots$

$W = \left(e^{i\frac{2\pi}{n}}\right)^6$



$n=4 \quad W^4 = 1$

$W = e^{2\pi i/4} = i$

powers $\rightarrow i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

the fourier matrix

making it orthonormal

divide by $\frac{1}{2}$

how to check if its cols are orthogonal

col 2 . col 3

$1 + i - 1 \mp i = 0$

col 2 . col 4

$(1)(1) + (-i)(-i) + (-1)(-1) + (i)(i)$

$1 - 1 + 1 - 1 = 0$

$F_4^H F_4 = I$

The key idea to FFT is that F_n is related to $F_{n/2}$

because $(W_{64})^2 = W_{32}$

$$\begin{bmatrix} F_{64} \\ {\scriptstyle 64\times 64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix}\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}\underbrace{\overset{P}{\begin{bmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & & 1 \end{bmatrix}}}_{\text{permutation matrix}} \qquad D = \begin{bmatrix} 1 & & & \\ & W & & \\ & & W^2 & \\ & & & \ddots & \\ & & & & W^{31} \end{bmatrix}$$

$\Leftarrow$ this breakup can be done

here we have

Lecture 27 Positive Definite Matrix (Tests)

Lets see 2 × 2 matrix tests

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

① $\lambda_1 > 0$, $\lambda_2 > 0$ → eigenvalues test

② $a > 0$, $ac - b^2 > 0$ → $|A|$ test

③ pivots $a > 0$ $\frac{ac-b^2}{a} > 0$

④ New test $x^TAx > 0$ except at $x = 0$

So how do we know ① and ② to pick out the ~~sa~~ matrix?

$\begin{bmatrix} 2 & 6 \\ 6 & ? \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 6 & 19 \end{bmatrix}$ but if we have the border line case $\begin{bmatrix} 2 & 2 \\ 6 & 18 \end{bmatrix} \rightarrow |A| = 0$

$? > 18$

$\lambda_1 = 0$, $\lambda_2 = 20$

pivots → 2

Graph of $f(x,y) = \vec{x}^T A \vec{x}$

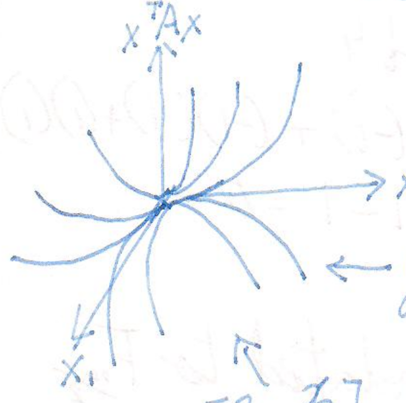
$= ax_1^2 + 2bx_1x_2 + cx_2^2$

$$\underset{x^T}{[x_1 \; x_2]} \underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$= [x_1 \; x_2] \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$$

$$= 2x_1^2 + 12x_1x_2 + 18x_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2 > 0 ? \text{ for all } x_1, x_2$$



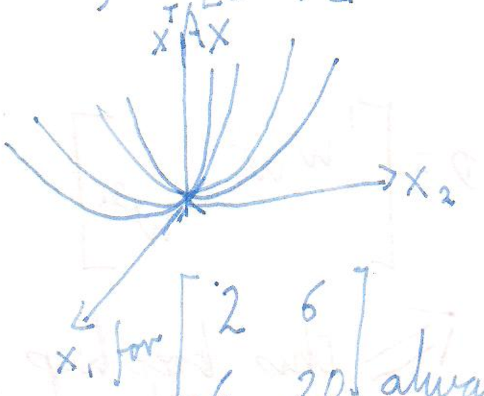
for $\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

---

in calculus min → $\frac{d^2u}{dx^2} \geq 0$

in linear algebra → matrix of 2nd derivatives is pos def.

$f(x_1 \dots x_n)$



for $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ always positive $2x_1^2 + 12x_1x_2 + 20x_2^2$

$= 2(x^2 + 6xx_2 + 10x_2^2)$

Note the eq. $\underbrace{2\cdot(x_1+3\cancel{y}x_2)}_{\text{this part is } \geq 0}{}^2+2x_2^2$ note it is $2x_1^2+12x_1x_2^2\,\underline{\underline{+18}}\,x_2^2$

---

lets do elimination

$$\begin{bmatrix} \textcircled{2} & 6 \\ 6 & 20 \end{bmatrix} \longrightarrow \overset{U}{\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}} \quad \overset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

pivot is $\underline{2}$, and we need to take $\underline{3}$ aways of row 1 from row 2

so $2(x_1+3x_2)^2+2x_2^2$

pivots

subtracting the rows

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

when do you have a min.
so this matrix needs to be positive

3x3 example

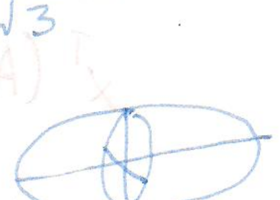
$$A = \begin{bmatrix} \boxed{2} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

det
$\underset{1\times1}{2}, \underset{2\times2}{3}, \underset{3\times3}{4}$ ......

pivots $2, \frac{3}{3}, \frac{4}{3}$

eigenvalues
$\lambda_1 = 2-\sqrt{2}, \lambda_2 = 2, \lambda_3 = 2+\sqrt{3}$

$x^TAx = 2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3 > 0$



The ellipsoid from this function wöuld have 3 axis
whose lengths are determined by the eigen values

LECTURE 28

A^TA is a positive definite matrix

$x^TAx > 0$ (except for $x=0$)

* PDMatrices usually comes from least squares

* Is the universe of a PDMatrix also PD?

$\lambda_1 \rightarrow A^{-1} \rightarrow \frac{1}{\lambda_1}$ since for PDMatrix have alway $\lambda_i > 0$, then A^{-1} would also be PDM

* If ~~*~~ A,B are pos def.

A+B is this pos def?

$\rightarrow x^T(A+B)x = x^TAx + x^TBx > 0$ Yes!

* If A is rectangular $m \times n$

It cant be pos. def but its not even symmetric

* Positive definitives is sort of positiveness concept in $\mathbb{R}$ for matrices

* For any A^TA, it is square, symmetric, pos.def.

This concept is like saying x^2 is always positive

$x^T(A^TA)x = (Ax)^T(Ax) = \|Ax\|^2$

how can I ensure that this 0 only when $x=0$

Then it should have no Null space. i.e. alls cols

Similar Matrices

Two square matrices $n \times n$, A and B are <u>similar</u> means: for some M

$$B = M^{-1} A M$$

Example

eigenvector matrix

$$S^{-1} A S = \Lambda \rightarrow$$ this says A is ~~singular~~ similar ~~to~~ with Λ

* This like putting matrices into families * where the best one Λ because a diagonal matrix with eigen values on the diagonal

Example : $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underset{M}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \underset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \underset{B}{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}$$

What is common between A & B. <u>They have the same eigenvalues</u> (In the above escample A, B <u>have eigenvalues $\lambda_i = 3, 1$</u>) They also have <u>the same number of eigenvectors</u>

proof $\rightarrow$ $Ax = \lambda x$

$$A M M^{-1} x = \lambda x$$

$$(M^{-1} A M) M^{-1} x = \lambda M^{-1} x$$

$$B \underline{M^{-1} x} = \lambda \underline{M^{-1} x}$$

hence B's eigenvalues remained the same, and

M^{-1}(eigvect of A)

Bad case $\lambda_1 = \lambda_2 = 4$ because we might not have two eigenvectors

one family $\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}$

$\begin{bmatrix}4 & 1\\0 & 4\end{bmatrix}$ are not in the same family although they both have the same eigenenvalues

This is only one matrix family

$$M^{-1}\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}M = 4M^{-1}IM = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}$$

This ~~is~~ family cannot have a diagonal matrix So this is the closest one and it is called the Jordan Form

Examples: More members of family $\begin{bmatrix}4 & 1\\0 & 4\end{bmatrix}$

$\begin{bmatrix}5 & 1\\-1 & 3\end{bmatrix}$, $\begin{bmatrix}4 & 0\\17 & 4\end{bmatrix}$, $\begin{bmatrix}a & \\ & 8-a\end{bmatrix}$ trace $= 8$ det $= 16$

$\left[\begin{array}{ccc|c}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\\hline 0 & 0 & 0 & 0\end{array}\right] \rightarrow \lambda_i = 0, 0, 0, 0$ and 2 eigenvectors, $\dim N(A) = 2$

$\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\\hline 0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{array}\right] \rightarrow \lambda = 0, 0, 0, 0$ and 2 eigenvectors

but these are not in the same family

these are not same because their Jordan blocks are not of the same size

Jordan block has 1 e-vector

Best J is Λ

$$J_i = \begin{bmatrix}\lambda_i & 1 & & 0\\ & \lambda_i & 1 & \\ & & \lambda_i & 1\\ 0 & & & \lambda_i\end{bmatrix}$$

LECTURE 29

SINGULAR VALUE DECOMPOSITION SVD

35

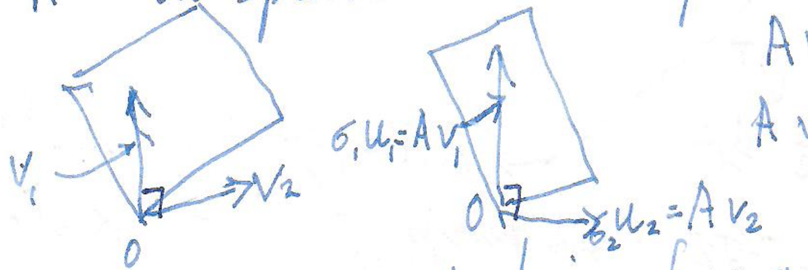
* decomposition / factorization

$$\underset{\text{any matrix}}{A} = \underset{\text{ortho normd}}{U} \ \underset{\text{diag}}{\Sigma} \ \underset{\text{orthonormal}}{V^T}$$

probably different

$A = S\Lambda S^{-1} \rightarrow$ if A is SPD $\rightarrow A = Q\Lambda Q^T \leftarrow$ story of positive definite

R^n row space column space R^m



$Av_1 = \sigma_1 u_1$

$Av_2 = \sigma_2 u_2$

v_1, v_2 are orthogonal basis in row space
u_1, u_2 gives orthogonal basis in col space

In SVD we are looking for an orthogonal basis is the row space which goes into the orthogonal basis in column space

$$A \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix}}_{\text{basis vectors in row space}} = \underbrace{\begin{bmatrix} u_1 & u_2 & \dots & u_r \end{bmatrix}}_{\text{basis vectors in col space}} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \ \sigma_r \end{bmatrix}$$

$AV = U\Sigma$

Example $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

v_1, v_2 in row space R^2?
u_1, v_2 in col space R^2?
$\sigma_1 > 0, \sigma_2 > 0$?

} need to find these

$AV = U\Sigma$

$A = U\Sigma V^{-1} = U\Sigma V^T$

$A^TA \rightarrow$ remember for PD $A = Q\Lambda Q^T$

$$\underbrace{A^TA} = V\Sigma^T \underbrace{\cancel{U^T}\cancel{U}} \Sigma V^T = V \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \ \sigma_n^2 \end{bmatrix} V^T$$

PD

* If there is some null space, we need to stick a basis for it in V ($v_{r+1} \dots v_n$) and for left null space U ($u_{r+1} \dots u_m$) and also extend Σ by 0s

→ A^TA → gives V in form of ~~its~~ AA^T eigenvectors

AA^T → gives U in form of A^TA eigenvectors ($AA^T = U\Sigma\Sigma^TU^T$)

E.g.

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \longrightarrow x_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \lambda_1 = 32$$

$$x_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \lambda_2 = 18$$

$$A = \underset{U}{\cancel{V}}\ \Sigma\ V^T$$

$$= \underset{U}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}} \overset{V^T}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}$$

← but there is sign ambigiuty

$$\longleftrightarrow AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda_1 = 32 \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda_2 = 18$$

E.g. 2

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \text{rank } 1$$

↗ row space $[4\ 3]$ ↗ multiples of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

↑ col space multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$N(A)$ — so directly $v_1 = \begin{bmatrix} .8 \\ .6 \end{bmatrix}$

$N(A^T)$ — directly $u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix} \quad \lambda_1 = 0, 125$$

so for $N(A) \rightarrow v_2 = \begin{bmatrix} .6 \\ -.8 \end{bmatrix}$, for $N(A^T) \rightarrow u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}_{U} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}}$$

(36)

* $v_1 \dots v_r$ is an orthonormal basis for the row space
$u_1 \dots u_r$ is an " " for the col. space
$v_{r+1}, \dots v_n$ is an " " for the null space
$u_{r+1}, \dots u_m$ is an " " for the $N(A^T)$

LECTURE 30

Linear Transformation T

Example 1: Projection

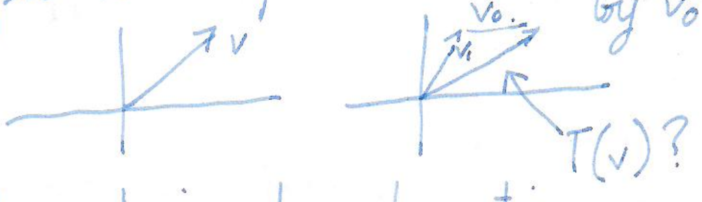
$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

every vector in ~~a plane~~ $\mathbb{R}^2$ to a line in $\mathbb{R}^2$ $(T(x)$

$$\left.\begin{aligned} T(v+w) &= T(v) + T(w) \\ T(cv) &= cT(v) \end{aligned}\right\} \quad T(cv+dw) = cT(v) + dT(w)$$

Also ~~not~~ note $T(0) = 0$

Example 2 : Shift whole plane by v_0



No because
$T(2v_1 + v_0) \neq 2T(v_1) + T(v_0)$

Not a linear transformation

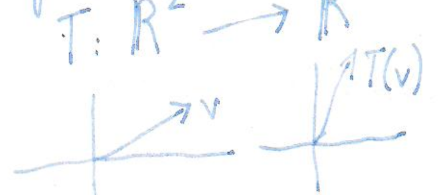
Ex~~map~~ample 3 : ~~Shift~~

$T(v) = \|v\|$, $T: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$T(-v) \neq -\|v\|$ So not a linear transformation

Example 4 : Rotation by 45°

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ Is a linear transformation



$T(v+w) = T(v) + T(w)$

Example 5: Matrix A

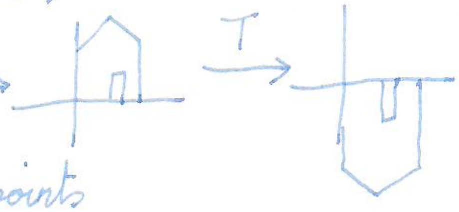
$T(v) = Av$ Is a linear transformation

$T(v+w) = A(v+w) = Av + Aw$ ✓

✓

$T(cv) = A(cv) = cA(v)$

Example $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow$



reverse y-axis points

Now how to get the matrix for such a linear transformation T

Example: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

like: $T(v) = Av$ ← 2 by 3 matrix

input in $\mathbb{R}^3$

output in $\mathbb{R}^2$

* Information needed to know $T(v)$ for all inputs?

If you know $T(v_1)$ and $T(v_2)$, if v_1 and v_2 were independent then I know a whole lot more i.e $T(cv_1 + cv_2)$ ← we know all of this

So ~~for~~ given a basis $v_1 \dots v_n$

Every $v = c_1 v_1 + \dots + c_n v_n$

Know $T(v) = c_1 T(v_1) + \dots + c_n T(v_n)$

} That is all we need to know for a transformation

* Coordinates come from a basis

$$v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \enclose{circle}{c_1}\, v_1 + \dots \enclose{circle}{c_n}\, v_2$$

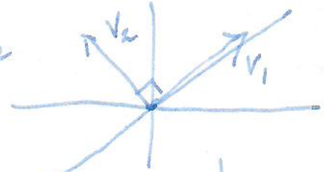
→ Construct a matrix A that represents a linear transformation T

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

1- Choose basis $v_1, \ldots, v_n$ for inputs $\mathbb{R}^n$ } now we
2- Choose basis $w_1, \ldots, w_n$ for output $\mathbb{R}^m$ } are working in coordinates

Want: matrix A does wat the lin. T does

Example: Projection



and we select the same basis for output $v_1 = w_1$

so we want

$T(v) = c_1 v_1 + c_2 v_2$

where $c_2 \Rightarrow 0$

that is removing the component perpendicular to v_1

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{\substack{\text{input} \\ \text{coords}}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\substack{\text{output} \\ \text{coord}}}{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}$$

eigenvector basis leads to diagonal matrix Λ

choosing diff bases

Proj. onto 45° line

use standard $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$

$$P = \frac{a a^T}{a^T a} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \quad \text{e.g.} \rightarrow P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \leftarrow 45°$$

Rule to find A. Given bases $v_1 \ldots\ldots v_n$
$w_1 \ldots\ldots v_n$

1st column of A : Write $T(v_1) = \textcircled{a_{11}} w_1 + \textcircled{a_{21}} w_2 + \ldots\ldots \textcircled{a_{m1}} w_m$

take 1st basis vector
apply transformation

2nd column of A : Write $T(v_2) = a_{12} w_1 + a_{22} w_2 + \ldots a_{m2} w_m$

$\vdots$

$$A \begin{pmatrix} \text{input} \\ \text{coords} \end{pmatrix} = \begin{pmatrix} \text{output} \\ \text{coords} \end{pmatrix}$$

e.g: $T = \frac{d}{dx}$ in: $c_1 + c_2 x + c_3 x^2 \rightarrow$ basis : $1, x, x^2$

derivative out: $c_2 + 2c_3 x \rightarrow$ basis : $1, x$

The derivatives are linear exactly because we can easily take derivatives by just knowing a few rules

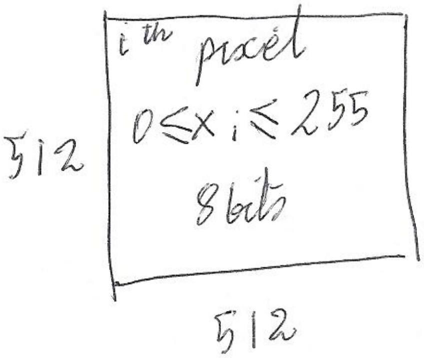
$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix} \longrightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

matrix for the derivative T

LECTURE 31

Change of basis

e.g. compression of images



$x \in \mathbb{R}^n$

$n = (512)^2$

standard compression JPEG

it is just about a change of basis

Standard basis

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \Rightarrow$$

Better basis

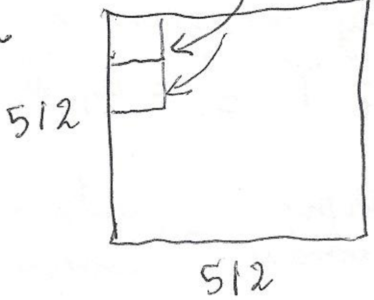
$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

this can give info about an image with no colour variation (DC vector)

can encode an image with half plane and another half

JPEG uses a Fourier basis 8 x 8

Change the basis for each 8x8 block individually



$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ W \\ W^2 \end{bmatrix}, \ldots\ldots$$

$$\begin{tikzcd}
\text{signal } x \arrow[d, "\text{change basis}", "\text{lossless}"'] \\
\text{coeffs } c \arrow[d, "\text{compression (throw away smaller coefficients by thresholding)}", "\text{lossy}"'] \\
\hat{c} \text{ (many zeros)} \arrow[d] \\
\hat{x} = \sum \hat{c}_i \underline{v_i}
\end{tikzcd}$$

~~output signal x~~

↖ basis vectors

Video : sequence of images which are extremely correlated

Wavelets $\mathbb{R}^8$

$$\underbrace{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix} \cdots\cdots}_{W_1 \quad W_2 \quad W_3 \quad W_4 \quad W_5 \ \cdots}$$

note that they are an orthogonal basis, hence $W^{-1} = W^T$

Now given a Pixel

$$\begin{bmatrix}P_1\\ \vdots\\ P_8\end{bmatrix} = c_1 W_1 + c_2 W_2 + \dots + c_8 W_8$$

changing the basis

$$P = \begin{bmatrix} \text{Wavelet basis matrix } W \end{bmatrix} \begin{bmatrix}c_1\\c_2\\ \vdots\\ c_8\end{bmatrix}$$

So $c = \boxed{W^{-1}} P$

good basis

1- fast multiplication (FFT for JPEG) (Fast Wavelet Transform)

2- Few basisvectors to represent the orig signal

Change of basis

W = new basis vectors

$$\underset{\text{old basis}}{\begin{bmatrix}x\end{bmatrix}} \longrightarrow \underset{\text{new basis}}{\begin{bmatrix}c\end{bmatrix}} \qquad \boxed{x = Wc}$$

Transformation $\leftrightarrow$ Matrix

T with respect to $v_1, \dots, v_8$
it has matrix A
with respect to $w_1, \dots w_8$
it has matrix B

would be the change of basis matrix

where A is similar to B $\rightarrow B = M^{-1}AM$

* What is A? Using basis $v_1 \dots v_8$
I know T completely from $T(v_1), T(v_2) \dots T(v_8)$, because every $x = c_1v_1 + c_2v_2 \dots + c_8v_8$

* Write $T(v_1) = a_{11}v_1 + a_{21}v_2 + \dots + a_{81}v_8$
$T(v_2) = a_{12}v_1 + a_{22}v_2 + \dots + a_{82}v_8$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{81} \\ \vdots & \vdots & & \\ a_{18} & a_{28} & & a_{88} \end{bmatrix}$$

* Now suppose we have an eigenvector basis
$T(v_i) = \lambda_i v_i$, What is A?

$$A = \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & \vdots & & \lambda_n \end{bmatrix}$$

LECTURE 32 Quiz review

Q1- $\frac{du}{dt} = Au = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} u$ find the general solution

$\rightarrow u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$

A is singular $\rightarrow \lambda_1 = 0,\ \lambda_2 = \sqrt{2}i,\ \lambda_3 = -\sqrt{2}i$

$-\lambda^3 - 2\lambda = 0$
$\lambda(\lambda^2 + 2) = 0$

Since A is antisymmetric $A^T = -A$, then the eigenvalues are pure imaginary

$u(t) = c_1 + c_2 e^{\sqrt{2}it} + c_3 e^{-\sqrt{2}it}$ $\rightarrow$ So its oscillating

Q- When would this solution comes back to its initial condition?

→ The eigenvectors of a (skew) symmetric matrix ~~have~~ are orthogonal. in fact it actually happens whenever $AA^T = A^TA$

Symm/ Skew Symmetric / orthogonal matrix ← true for

Q2 $\lambda_1 = 0$, $\lambda_2 = c$, $\lambda_3 = 2$

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ Now what is the matrix?

(a) all c's that make the matrix is diagonalizable?
the eigenvectors are orthogonal — hence for all c it will be diagnolizable

(b) symmetric?
c needs to be real, for the matrix to ~~to~~ be symmetric

(c) positive definite?
is a sub-case of symmetric. So first condition is that c needs to be real. For PD we need all $\lambda_K > 0$, ~~then~~ since we have $\lambda_1 = 0$ its not PD

(d) markov matrix?
one eigenvalue is always 1, and others are less than <1. So cant be markov matrix.

(e) ~~1 half of A~~ $\frac{A}{2}$ be projection matrix?

eigenvalues of projection matrices $\lambda = 0$ ~~or~~ 1.
So we have $P = \frac{A}{2} \rightarrow \lambda_1 = 0$, $\lambda_2 = \frac{c}{2}$, $\lambda_3 = 1$.
So need $c = 0$ or 2.

Q3 SVD $U \Sigma V^T$

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$$

→ if $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ then A is singular, the rank = 1, the dim of N(A) is 1

a vector $\overset{\text{the}}{\text{is}}$ ~~the~~ N(A) will be v_2

Ⓠ4 Given A is symmetric and orthogonal

What could be the possible eigenvalues -

A is symmetric → λ is real

A is orthogonal → $|\lambda| = 1$ →

Why?

$Qx = \lambda x$

$\|x\| = |\lambda| \|x\|$

because Q is like a rotation matrix, doesnt change length

a could be PD? → no

b is it diagnolizable? →

yes for all symmetric and/or orthogonal matrix

c could it be non-singular?

of course, look at the eigenvalues

d Prove $\frac{1}{2}(A+I)$ is a projection matrix?

$P^2 \overset{?}{=} P \longrightarrow \frac{1}{2}(A^2+2A+I) \overset{?}{=} \frac{1}{2}(A+I)$

so wat is A^2 .... we are given $A = A^T = A^{-1}$

so ~~AA~~ $A^2 = I$

$\frac{1}{2}(2A+2I) = A+I$ · so yes!

LECTURE 33

2- sided inverse

$AA^{-1} = I = A^{-1}A$

$r = m = n$ (square matrix, full rank)

Left inverse

0 or 1 solns to $Ax = b$

A^TA and the rank $= n$

$$\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}} \longrightarrow (A^TA)^{-1}A^TA = I$$

$$\underset{n \times m}{A^{-1}_{left}} \underset{m \times n}{A} = \underset{n \times n}{I}$$

* If the matrix is rectangular there is either a left or right inverse ~~cat~~ cause there got to be some free vars (null space)

Right-~~inv~~ inverse

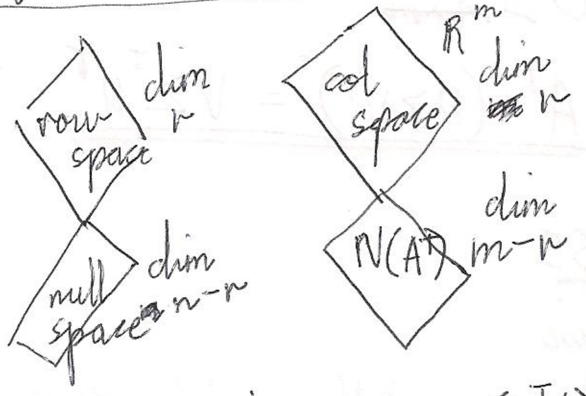
full row rank $r = m < n$ indep rows

$N(A^T) = \{0\}$

$$\boxed{\begin{array}{c}\infty \text{ solns to } Ax = b \\ n-m \text{ free variables}\end{array}}$$

$$\underbrace{A^T(AA^T)^{-1}}_{A^{-1}_{right}} \longrightarrow AA^T(AA^T)^{-1} = I$$

$$\underset{m \times n}{A} \ \underset{n \times m}{A^{-1}_{right}} = \underset{m \times m}{I}$$



* What is $AA^{-1}_{left} = A\underbrace{(A^TA)^{-1}A^T}_{\text{its like a projection onto the col. space}}$ $\quad A^{-1}_{right}A = \underbrace{A^T(AA^T)^{-1}}_{\text{its a projection onto the row sp.}}A$

* Mapping from row space to col. space is a 1-1

~~If x in row space, y in col. space then~~

If x, y is in the row space then $\underbrace{Ax \neq Ay}$ (if $x \neq y$)

Proof Suppose $Ax = Ay$

$$A(x-y) = 0$$

So $x-y$ needs to be in the null space. Plus since x, y were in the row space, $x-y$ needs to be in the row space. Hence if $x-y$ needs to the zero vector

* Find the pseudo-inverse A^+

$$\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots \ \sigma_r & \\ 0 & & 0 \end{bmatrix} \begin{matrix} \text{rank } r \\ m \text{ rows} \\ m \times n \end{matrix}$$

n cols

1st way From SVD $\to A = U\Sigma V^T$

So $\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots \ \frac{1}{\sigma_r} & \\ 0 & & 0 \end{bmatrix}_{n\times m}$

$$\Sigma\Sigma^+ = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}_{m\times m}$$

$$\Sigma^+\Sigma = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}_{n\times n \ \cancel{m\times m}}$$

$$A = U\Sigma V^T \to \underline{\underline{A^+ = (U\Sigma V^T)^+ = V\Sigma^+ U^T}}$$

LECTURE 34

REVIEW COURSE

Q - Given ~~3×n~~ m×n matrix

$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ no soln. → tells us its rank defficient $r<m$

$Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has 1 soln. → $N(A) = \{0\}$ so $r = n$

$\to m = 3 > n = r$

e.g. matrix $A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

→ A^TA is invertible ?

→ AA^T is pos. definite ?

~~A~~ but AA^T is 3x3 for A_{3x2} (second example)

3. but AA^T rank would be 2 . So False

→ det A^TA = det AA^T ?

is false . This would be true if A is square

→ $A^Ty = c$, ~~poor~~ atleast 1 soln. for every c
in fact ∞ soln. for every c ?

$A^T_{n\times m}$ so full row rank (n) would always have a soln. This is true because the original matrix had n independent cols.

dim $N(A^T) = m - n > 0$, so ∞ soln.s because m-n free variables

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Q- $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

→ Solve $Ax = v_1 - v_2 + v_3$? $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

→ Suppose $v_1 - v_2 + v_3 = 0$, so x is unique ?

So the question is if the soln is unique then there would be nothing in the N(A). But we know here $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in the null space so False

→ v_1, v_2, v_3 are orthonormal ?

What combination of v_1 & v_2 closest to v_3 ?

$a_1 v_1 + a_2 v_2$ = closest to v_3 . All a_1 & $a_2 = 0$ because they are orthonormal

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Q - Markov matrix

$$A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}$$

col 1 + col 2 = 2 (cols 3)

→ So what are the λ?

Since singular $\lambda_1 = 0$, $\underbrace{\lambda_2 = 1}_{\text{because Markov mat}}$, $\lambda_3 = -.2$

→ Suppose we start the markov process $u_k = A^k u_{k-1}$

$u_{(0)} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$ ... what is the steady state?

$$u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + c_3 \lambda_3^k x_3$$

$$= \cancel{c_1 x_1} \; c_2 x_2 + c_3 (-.2)^k x_3$$

So for steady state $u_\infty = c_2 x_2$

$$A - \lambda_2 = \begin{bmatrix} -.8 & .4 & .3 \\ .4 & -.8 & .3 \\ .4 & .4 & -.6 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow u_\infty = c \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

$$u_\infty = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -.8 & .4 & .3 \\ 0 & -.6 & .45 \\ 0 & .6 & -.45 \end{bmatrix} \longrightarrow \begin{bmatrix} -.8 & .4 & .3 \\ 0 & -.6 & .45 \\ 0 & 0 & 0 \end{bmatrix}$$

Q - Project onto $a = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$? $P = \dfrac{a a^T}{a^T a}$

Q - $\lambda_1 = 0$, $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\lambda_2 = 3$, $x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A = S \Lambda S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

Q - A real matrix that can't be factored $B^T B$ for any B?

$A \neq B^T B$ for any B

So any non-symmetric A, it will be true

→ orthogonal ~~eigen~~ eigenvectors matrix A but not symmetric matrices ?

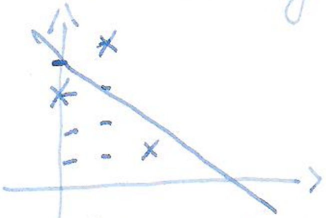
matrix could be skew symmetric $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

or orthogonal matrix $\begin{bmatrix}\cos & -\sin\\ \sin & \cos\end{bmatrix}$ all these matrix what have complex ~~eigenvalues~~ eigenvectors

Q- $\begin{bmatrix}1 & 0\\ 1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\hat{c}\\ \hat{d}\end{bmatrix} = \begin{bmatrix}3\\ 4\\ 1\end{bmatrix}$ get the least square soln $\begin{bmatrix}\hat{c}\\ \hat{d}\end{bmatrix} = \begin{bmatrix}11/3\\ -1\end{bmatrix}$

→ What is P (proj of b onto C(A)) is $\frac{11}{3}$(col 1) ~~+~~ -1(col 2)

→ Draw the straight line



→ Find a different b so the soln is 0 ?

$\begin{bmatrix}\hat{c}\\ \hat{d}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \longrightarrow b = \begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$ just find orthogonal vector to the two columns

THE END!